Sequences with identical autocorrelation spectra

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Supported by

- National Science Foundation Awards 1815487 and 2206454
- Ramanujan Research Scholarship (A. Rahman) and Efrem Ostrow Scholarship (M. J Ward), Department of Mathematics, California State University, Northridge
- Open Science Grid Consortium (computational resources), funded by National Science Foundation awards 2030508 and 1836650

Selmer Center/Simula UiB Seminar

University of Bergen
20 September 2024

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We are interested in the discrete, one-dimensional (sequences) analogue of this problem.

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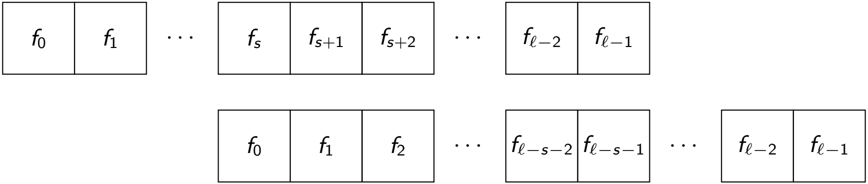

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For a binary sequence of length ℓ, since our $f_0, \ldots, f_{\ell-1}$ are in $\{1, -1\}$ (and all other $f_j = 0$), we have

$$C_f(0) = \sum_{j=0}^{\ell-1} |f_j|^2 = \ell.$$

Laurent polynomials on the unit circle

Identify sequences with polynomials:

$$f = (f_0, \ldots, f_{\ell-1}) \leftrightarrow f(z) = f_0 + f_1 z + \cdots + f_{\ell-1} z^{\ell-1} \in \mathbb{C}[z].$$

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These subfields are self-conjugate (closed under conjugation).

Associate Laurent polynomials

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Are there any other ways in which sequences could be equicorrelational?

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Trivial equicorrelationality is an equivalence relation that is coarser than associateness, and the trivial equicorrelationality class of f in $\mathbb{C}[z, z^{-1}]$ is written $[\![f]\!] = [f] \cup [\overline{f}]$.

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Is trivial equicorrelationality the only kind of equicorrelationality?

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These two binary sequences of length 9 are equicorrelational and are not associates nor is one the associate of the other's conjugate (reverse):

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Both have autocorrelation spectrum

$$z^{-8} - 3z^{-6} + z^{4} - 3z^{2} + 9z^{0} - 3z^{2} + z^{4} - 3z^{6} + z^{8}, \text{ i.e.,}$$

$$C_f(s) = C_g(s) = \begin{cases} 9 & \text{if } s = 0 \\ -3 & \text{if } s \in \{\pm 2, \pm 6\} \\ 1 & \text{if } s \in \{\pm 4, \pm 8\} \\ 0 & \text{otherwise.} \end{cases}$$

Equivalence relations

Relation	Class	Class restricted to $F[z, z^{-1}]$
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A generalized palindrome is an $f \in \mathbb{C}[z, z^{-1}]$ such that $\overline{f}$ is associate to f: then $[f] = [\overline{f}]$ so $[\![f]\!] = [f]$.

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A generalized palindrome is an $f \in \mathbb{C}[z, z^{-1}]$ such that $\overline{f}$ is associate to f: then $[f] = [\overline{f}]$ so $[\![f]\!] = [f]$.

Binary examples: f is a palindrome, g is an antipalindrome

$f =$

+	−	−	−	−	+

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Equivalence relations

Relation	Class	Class restricted to $F[z, z^{-1}]$
Associate	$[f]$	$[f]_F$
Trivial equicorrelational	$[\![f]\!] = [f] \cup [\overline{f}]$	$[\![f]\!]_F = [f]_F \cup [\overline{f}]_F$
Equicorrelational	$[\![\![f]\!]\!]$	$[\![\![f]\!]\!]_F$

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Self-conjugate associate classes are entirely made of generalized palindromes; other associate classes have none.

Main theorem

F is a self-conjugate subfield of $\mathbb{C}$ and factorize $f \in F[z, z^{-1}] \setminus \{0\}$

$$f(z) = u f_1^{a_1} \cdots f_m^{a_m} g_1^{b_1} \cdots g_n^{b_n} \overline{g_1}^{c_1} \cdots \overline{g_n}^{c_n}$$

with u a unit, the rest $F[z, z^{-1}]$-irreducibles, with $f_1, \ldots, f_m$ being generalized palindromes.

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$$\llbracket f \rrbracket_F = \bigcup_{\substack{b', c' \in \mathbb{N}^n \\ b' + c' = b + c}} \left[f_1^{a_1} \cdots f_m^{a_m} g_1^{b'_1} \cdots g_n^{b'_n} \overline{g_1}^{c'_1} \cdots \overline{g_n}^{c'_n} \right]_F$$

$$= \bigcup_{\substack{b', c' \in \mathbb{N}^n \\ b' + c' = b + c \\ b' \leq c'}} \left\llbracket f_1^{a_1} \cdots f_m^{a_m} g_1^{b'_1} \cdots g_n^{b'_n} \overline{g_1}^{c'_1} \cdots \overline{g_n}^{c'_n} \right\rrbracket_F .$$

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- The sequence f is nontrivially equicorrelational to some other sequence in $F[z,z^{-1}]$ if and only if $\prod_{j=1}^n (b_j + c_j + 1) \geq 3$.

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8230 [2] + 16 [4] means 8230 classes of volume 4 and 16 of vol. 2

len.	nontriv.	len.	nontriv.	len.	nontriv.
1		16	12 [2]	31	26 [2]
2		17	1 [2]	32	1136 [2]
3		18	42 [2]	33	1105 [2]
4		19		34	30 [2]
5		20	44 [2]	35	349 [2]
6		21	67 [2]	36	8230 [2] + 16 [4]
7		22		37	
8		23		38	
9	1 [2]	24	422 [2]	39	4102 [2]
10		25	36 [2]	40	6288 [2]
11		26		41	4[2]
12	8 [2]	27	348 [2] + 1 [4]	42	17574 [2]
13		28	180 [2]	43	22 [2]
14		29		44	3104 [2]
15	14 [2]	30	1214 [2]		

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Open Question 2: Count the number E_n of equivocal binary sequences of length n. Does $E_n/2^n$ tend to 0 as $n \to \infty$?