On the normality of Boolean quartics

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Walk along Claude’s gardens

Boolean function

Claude's favorite universe is made up of an infinite number of gardens $B(1)$, $B(2)$, $B(3)$ ... The flowers of the garden $B(m)$ are Boolean functions often called m-bit functions. With friends, we like to spend time in garden $B(8)$, searching for nice object among 8-bit functions...

115792089237316195423570985008687907853269984665640564039457584007913129639936 flowers...

Rare pearl

What is the minimal linearity of a balanced Boolean function?

Walsh coefficient

spectral amplitude

$$\widehat{f}(a) = \sum_{x \in \mathbb{F}_2^m} (-1)^{f(x)+a.x} \qquad R(f) = \max_{a \in \mathbb{F}_2^m} |\widehat{f}(a)|$$

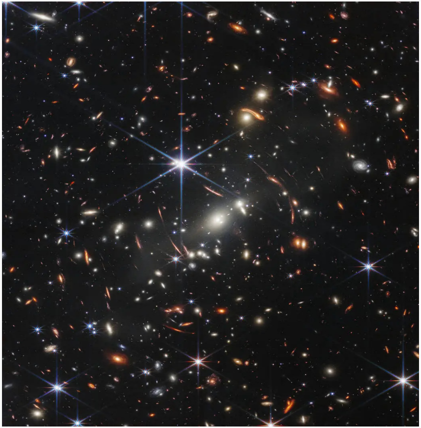
$$\mathrm{R}(m) = \min_{f} R(f) \qquad \mathrm{R_b}(m) = \min_{\widehat{f}(0)=0} R(f)$$

spectral radius

balanced radius

galaxy cluster SMACS 0723 : 2^{32} light years . . .

NASA's James Webb Space Telescope has produced the deepest and sharpest infrared image of the distant universe to date.



The number of atoms in the universe is estimated to 10^{80}

The number of 8-bit functions is $\approx 10^{77}$

Open question regarding garden $B(8)$

All is known for Boolean function spaces in dimension smaller than 7, below is a list of open questions are in dimension 8 :

- normality of bent functions ?
- classification of bent functions ?
- normality of quartics ?
- covering radius of $RM(2, 8)$?
- covering radius of $RM(3, 8)$?
- ~~covering radius of $RM(4, 8)$?~~
- linearity of balanced functions ?
- *What else... Let me know!*

Numerical point of view

For all these questions, in dimension 8, **numerical approaches** based on **classifications under the action of the affine general group** may give results.

Classification in dimension 7 and 8

$\mathrm{AGL}(m,2)$ acts naturally on Reed-Muller codes, RM-spaces :

$$B(s,t,m) := \{ \sum_{s \leq |S| \leq t} a_S X_S \quad \} = RM(t,m)/RM(s-1,m)$$

$\widetilde{B}(s,t,m)$ denotes a system of represensentatives of $B(s,t,m)$

$s \backslash t$	1	2	3	4	5	6	7
0	3	12	**3486**	$10^{13.5}$	$10^{19.8}$	$10^{21.9}$	$10^{22.2}$
1	2	8	1890	$10^{13.1}$	$10^{19.5}$	$10^{21.6}$	$10^{21.9}$
2		4	179	$10^{11.0}$	$10^{17.3}$	$10^{19.5}$	$10^{19.8}$
3			12	68443	$10^{11.0}$	$10^{13.1}$	$10^{13.5}$
4				12	179	1890	**3486**
5					4	8	12
6						2	3
7							2

Class numbers of $B(s,\ t,\ 7)$

$s \backslash t$	1	2	3	4	5	6	7	8
1	2	9	3814830	$10^{27.6}$	$10^{44.5}$	$10^{52.9}$	$10^{55.3}$	$10^{55.6}$
2		5	**20748**	$10^{25.2}$	$10^{42.0}$	$10^{50.5}$	$10^{52.9}$	$10^{53.2}$
3			32	$10^{16.7}$	$10^{33.6}$	$10^{42.0}$	$10^{44.5}$	$10^{44.8}$
4				**999**	$10^{16.7}$	$10^{25.2}$	$10^{27.6}$	$10^{27.9}$
5					32	20748	3814830	7611801
6						5	9	14
7							2	3
8								2

Class numbers of $B(s,\ t,\ 8)$

see our recent works [7] with Valérie Gillot.

Normality, relative degree

Normality, weak normality

The notion of normality was introduced by Hans Dobbertin (1994) in order to produce the inequality :

$$\mathrm{R_b}(2t) \leq 2^t + \mathrm{R_b}(t) \Longrightarrow \mathrm{R_b}(8) \leq 24$$

Pascale Charpin **(2004)**

A Boolean function $f \in B(m)$ is said to be **normal** if there exists a subspace V of $\mathbb{F}_2^m$ with middle dimension $\lceil m/2 \rceil$ such that f is constant on some translate $a + V$ with $a \in \mathbb{F}_2^m$.

It is convenient to use the notation

$$f_{a,V} : v \mapsto f(v+a), \quad f_{a,V} \in B(V) \sim B(\dim V)$$

weak normality

A non normal f is **weakly normal** when $f_{a,V}$ is affine for some $a + V$ with middle dimension.

relative degree

The degree of $f_{a,V}$ is called the **relative degree** of f on $a + V$:

$$\deg_{a+V}(f) := \deg(f_{a,V}), \qquad \text{(by convention} \geq 0)$$

Definition

The r-**degree** of f is the minimal relative degree of f for all affine spaces $a + V$, where $\dim(V) = r$.

$$0 \leq \deg_r(f) = \min\{\deg_{a+V}(f) \mid \dim V = r \quad \text{and} \quad a \in \mathbb{F}_2^m\}.$$

Notions of normality translate:

$$\deg_{\lceil m/2 \rceil}(f) = \begin{cases} 0, & f \text{ is normal;} \\ 1, & f \text{ is weakly normal;} \\ \geq 2, & f \text{ is abnormal.} \end{cases}$$

combinatorial parameters

For integers $r \leq m$ and $k \leq m$, we explore :

$$D_r(k, m) = \max\{\deg_r(f) \mid \deg(f) \leq k\}.$$

In a recent note (August 2024), Jan Kristian Haugland studies :

$$g(m, r) = \max_f \deg_r(f) = \max_k D_r(k, m)$$

Theorem

If $r > t \geq 0$ *and* $m \geq 2^{r-1} + r - \lfloor 2^{t-1} \rfloor$ *then* $g(m, r) \leq t$

Conjecture (Haugland)

If $r \geq 2$ *then*

$$g(r+2, r) = r - 2$$

$$
\begin{gathered}
f(x) = x_8x_7x_5 \oplus x_8x_6x_5 \oplus x_8x_4 \oplus x_8x_6x_4 \oplus x_7x_6x_4 \oplus x_8x_7x_6x_4 \oplus x_8x_5x_4 \oplus x_7x_5x_4 \oplus \\
x_8x_7x_5x_4 \oplus x_8x_6x_5x_4 \oplus x_8x_7x_6x_5x_4 \oplus x_7x_3 \oplus x_8x_7x_3 \oplus x_8x_7x_6x_3 \oplus x_7x_5x_3 \oplus x_6x_5x_3 \oplus \\
x_8x_6x_5x_3 \oplus x_7x_6x_5x_3 \oplus x_8x_7x_6x_5x_3 \oplus x_7x_4x_3 \oplus x_6x_4x_3 \oplus x_7x_6x_4x_3 \oplus x_8x_7x_6x_4x_3 \oplus \\
x_8x_5x_4x_3 \oplus x_7x_5x_4x_3 \oplus x_8x_7x_5x_4x_3 \oplus x_8x_6x_5x_4x_3 \oplus x_7x_6x_5x_4x_3 \oplus x_8x_7x_6x_5x_4x_3 \oplus x_8x_7x_2 \oplus \\
x_6x_2 \oplus x_8x_6x_2 \oplus x_7x_6x_2 \oplus x_8x_7x_6x_2 \oplus x_8x_7x_5x_2 \oplus x_7x_6x_5x_2 \oplus x_8x_7x_6x_5x_2 \oplus x_8x_4x_2 \oplus x_6x_4x_2 \oplus \\
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x_7x_5x_1 \oplus x_8x_7x_5x_1 \oplus x_6x_5x_1 \oplus x_8x_6x_5x_1 \oplus x_7x_6x_5x_1 \oplus x_8x_7x_6x_5x_1 \oplus x_8x_4x_1 \oplus x_8x_7x_4x_1 \oplus \\
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x_8x_6x_5x_3x_1 \oplus x_4x_3x_1 \oplus x_8x_4x_3x_1 \oplus x_8x_7x_4x_3x_1 \oplus x_6x_4x_3x_1 \oplus x_8x_6x_4x_3x_1 \oplus x_7x_6x_4x_3x_1 \oplus \\
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x_7x_6x_5x_2x_1 \oplus x_8x_7x_6x_5x_2x_1 \oplus x_4x_2x_1 \oplus x_8x_4x_2x_1 \oplus x_7x_4x_2x_1 \oplus x_8x_7x_4x_2x_1 \oplus x_7x_6x_4x_2x_1 \oplus \\
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x_7x_3x_2x_1 \oplus x_8x_7x_3x_2x_1 \oplus x_6x_3x_2x_1 \oplus x_8x_6x_3x_2x_1 \oplus x_7x_6x_3x_2x_1 \oplus x_8x_7x_6x_3x_2x_1 \oplus \\
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x_8x_7x_4x_3x_2x_1 \oplus x_6x_4x_3x_2x_1 \oplus x_7x_6x_4x_3x_2x_1 \oplus x_5x_4x_3x_2x_1 \oplus x_8x_5x_4x_3x_2x_1 \oplus x_6x_5x_4x_3x_2x_1
\end{gathered}
$$

In her PhD thesis, Sylvie Dubuc presented the first example of a **non-normal** function in $B(8)$. It has degree 6 comprising 140 monomials.

We checked

it is **weakly normal**!

$$bcdegh + bcdeh + bcdgh + bcdf + bcef + bcdh + bdfh + begh + abc + cde + bef + bdg + cdg + deh + cfh + cgh + bc + be + ce + df + bg + ch + gh + a + b$$

Haugland table

r \ m	1	2	3	4	5	6	7	8	9	10	11	12
1	1	0	0	0	0	0	0	0	0	0	0	0
2		2	1	0	0	0	0	0	0	0	0	0
3			3	2	1	0	0	0	0	0	0	0
4				4	3	2	2	≥ 1	≥ 0	≥ 0	0 or 1	0
5					5	4	3	3	≥ 2	≥ 2	≥ 0	≥ 0
6						6	5	≥ 4	≥ 3	≥ 3	≥ 3	≥ 2

$$g(m, r) = \max_{f \in B(m)} \deg_r(f)$$

Question 1

Does there exist a Boolean function $B(8)$ with 4-relative degree 2?

Does there exist a non normal bent function in $B(m)$?

Asked by Hans Dobbertin 1994...

Sylvie Dubuc

All members of $B(6)$ are normal!

(2001)

Anne Canteaut, Magnus Daum, Hans Dobbertin, Gregor Leander

Some Kasami power functions are 14-bit abnormal bent functions.

(2006)

Gary McGuire, Gregor Leander

Applying magic tricks to Kasami functions, provided 10-bit and 12-bit abnormal bent.

(2009)

Normality of bent functions in $B(8)$

Sylvie Dubuc **(2001)**

All bent cubics of $B(8)$ are normal!

Pascale Charpin **(2004)**

Does it exist non-normal bent functions of 8 variables and degree 4?

Last BFA conference, Luca Mariot, Stjepan Picek and Alexandr Polujan, exhibited example of (non normal) weakly normal function in $B(8)$ coming from the partial spread class.

Question 2

Are there any abnormal 8-bit bent functions?

Numerical facts on relative degree

$D_r^\ddagger(k, 6)$ illustration of Dubuc's result

$r \backslash k$	1	2	3	4	5	6
5	0	2	3	4	3	4
4	0	2	2	2	2	2
3	0	0	0	0	0	0
2	0	0	0	0	0	0

Maximal r-relative degree of functions of degree k of $B(6)$.

Sylvie Dubuc

All flowers of $B(6)$ are normal!

Remark

If $f \in B(7)$ *or* $f \in B(8)$ *then* $\deg_4(f) \leq 2$.

4-relative degree in $B(7)$?

The following member of $RM(6,7)$ has 4-relative degree 2 :

$$cd+abcd+ce+ade+acde+bcde+abcde+abf+adf+bdf+cdf+abdef$$
$$+cdef+acdef+ag+abg+cg+acdg+eg+aceg+bceg+deg+adeg$$
$$+abdeg+cdeg+abcdeg+fg+afg+abfg+adfg+bdfg+abcdfg+befg+bcefg$$

So,

$$D_4(7)=2$$

See Jan Kristian Haugland note on arXiv.

$D_r^\dagger(k,7)$

$r\backslash k$	1	2	3	4	5	6	7
6	0	2	3	≥ 4	≥ 4	5	≥ 4
5	0	2	3	≥ 3	≥ 3	≥ 3	≥ 3
4	0	1	1	≥ 1	≥ 1	2	≥ 1
3	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0

Maximal r-relative degree of functions of degree k of $B(7)$.

Numerical fact using the 3486 members of $\widetilde{B}(0,3,7)$

All cubics in $B(7)$ are normal or weakly normal.

The minoration in the right part of the table $D_r^\dagger(k,7)$ were obtained at random, by adding a random cubic to the 3486 members of $\widetilde{B}(4,7,7)$.

Conjecture

All the quartics of $B(7)$ are normal or weakly normal.

Numerical facts on 8-bit functions

8-bit cubics

Numerical fact

All the cubics in $B(8)$ are normal or weakly normal.

Proof.

For $f \in RM(3,8)$ and for any hyperplane $H \subset \mathbb{F}_2^8$, the restriction of f to H is a cubic in $B(H) \sim B(7)$. $\square$

Alternatively, it is a numerical fact using the 20748 members of $\widetilde{B}(2,3,8)$.

$r \backslash \deg_r$	0	1	2	3
7	10	0	53	20 712
6	130	21	1 910	18 714
5	5 504	5 227	10 044	0
4	20 748	0	0	0

Distribution of relative degrees of 20748 cubics of $\widetilde{B}(2,3,8)$

Question 2

Are there any abnormal 8-bit bent functions?

possible approaches :

- harmonic analysis folklore
- classification of bent functions

Xiang-Dong Hou condition

The ANF of an 8-bit bent function

$$f = \sum_{2 \leq |S| \leq 4} a_s X_S = h + c + q = \begin{cases} h, \text{quartic;} \\ c, \text{cubic;} \\ q, \text{quadric.} \end{cases}$$

satisfies a system of 36 quadratic equations parametrized by subset $W \subseteq \{1, 2, \ldots, 8\}$ of cardinality 8, 7 and 6 :

$$\sum_{\substack{\{S, S'\} \\ S \cup S' = W}} a_S a_{S'} = 0$$

Given h, the cubic part depends on a linear system of 56 unknowns and 8 equations and one use the <u>stabilizer</u> of h to build a part of cover set.

cover set

Adapting the counting method presented with Gregor Leander in 2011, we obtain a **cover-set** of 8-bit of the set of bent functions of degree 4 :

- size $355\,073\,617 \approx 2^{28.52}$;
- approx 100 cores during 2-3 months (night-time).

Data and bent functions are available :

langevin.univ-tln.fr/project/

It is an easy task to check the weak normality of all these functions.

Numerical fact using the 999 members of $\widetilde{B}(4,4,8)$

All 8-bit **bent** functions are normal or weakly normal.

Conclusion

Practice $\rightsquigarrow$ theory

Now, thanks to our numerical approach, we know that it is true that all 8-bit bent functions are normal...

Il est plus facile de démontrer une conjecture lorsqu'on sait d'une manière ou d'une autre qu'elle est vraie.

It's easier to prove a conjecture when you know one way or another that it's true.

to be continued!

Jacques Hadamard

Summary

Main result

The construction of a **small** cover set of 8-bit bent functions

[forthcoming classification soon]

Milestone for $B(8)$

All 8-bit **bent** functions are normal or weakly normal.

Conjecture

All 7-bit **quartics** *are normal or weakly normal.*

Conjecture

All 8-bit **quartics** *are normal or weakly normal.*

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Hidden motivation

Hidden motivation

We recently construct millions of CCZ class of quadric APN in dimension 8 considering extensions of a $(8,4)$-bent vectorial functions.

Let β be the bent indicator of $F\colon \mathbb{F}_2^m \longrightarrow \mathbb{F}_2^m$,

$$\forall b \in \mathbb{F}_2^m, \quad \beta(b) = 1 \iff \text{component } b.F \text{ is bent.}$$

Conjecture (kind)

The bent indicator of a quadratic APN in 8 variables is normal.

In the space of homogeneous quadratic forms of 8 variables, a space of dimension 28, the indicator of the set of bent functions is a **quartic**...