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Quantum Codes from Classical Codes

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Overview

- qubits and qudits
- quantum codes (QECC)
  - CSS codes
  - stabilizer codes
- entanglement assisted quantum codes (EAQECC)
  - general code constructions
  - varying the hull dimension
  - propagation rules
- summary & outlook

Quantum Information

Quantum-bit (qubit)

basis states:

$$\text{“0”} \mathrel{\hat{=}} |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2, \qquad \text{“1”} \mathrel{\hat{=}} |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$$

general state:

$$|q\rangle = \alpha|0\rangle + \beta|1\rangle \qquad \text{where } \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1$$

measurement (read-out):

result “0” with probability $|\alpha|^2$

result “1” with probability $|\beta|^2$

Quantum Information

Quantum register

basis states:

$$|b_1\rangle \otimes \ldots \otimes |b_n\rangle =: |b_1 \ldots b_n\rangle = |\boldsymbol{b}\rangle \qquad \text{where } b_i \in \{0,1\}$$

general state:

$$|\psi\rangle = \sum_{\boldsymbol{x}\in\{0,1\}^n} c_{\boldsymbol{x}} |\boldsymbol{x}\rangle \qquad \text{where } \textstyle\sum_{\boldsymbol{x}\in\{0,1\}^n} |c_{\boldsymbol{x}}|^2 = 1$$

$\longrightarrow$ normalized vector in $(\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^{2^n}$

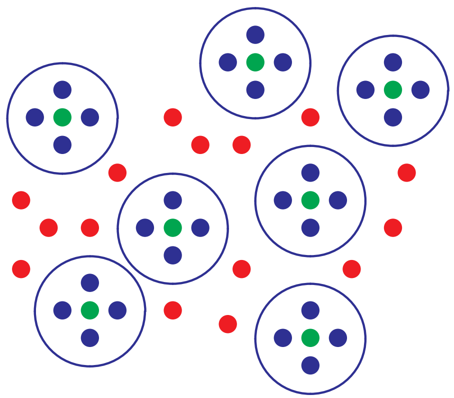
Qudits

generalization to $(\mathbb{C}^q)^{\otimes n}$: basis states $|\boldsymbol{b}\rangle$ labelled by vectors $\boldsymbol{b} \in \mathbb{F}_q^n$
$\Longrightarrow$ group algebra $\mathbb{C}[\mathbb{F}_q^n]$

The Basic Idea of QECC

Classical Codes

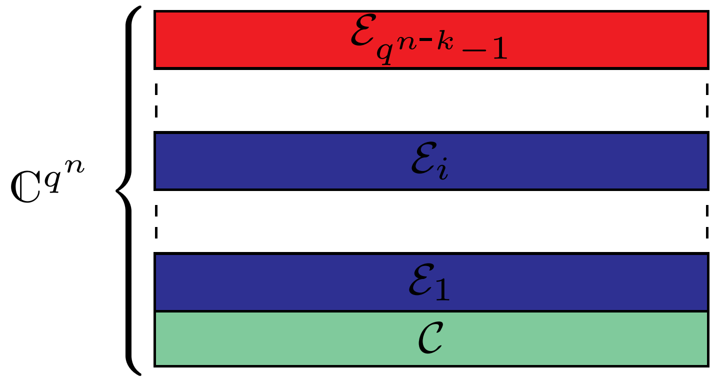
Partition of the set of all words of length n over an alphabet of size q.



- codewords
- errors of bounded weight
- other errors

Quantum Codes

Orthogonal decomposition of the vector space $\mathcal{H}^{\otimes n}$, where $\mathcal{H} \cong \mathbb{C}^q$.



$$\mathcal{H}^{\otimes n} = \mathcal{C} \oplus \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_{q^{n-k}-1}$$

encoding: $|\boldsymbol{x}\rangle \mapsto U_{\text{enc}}(|\boldsymbol{x}\rangle \otimes |0\rangle)$

Quantum Errors

Bit-flip error:

- Interchanges $|0\rangle$ and $|1\rangle$. Corresponds to "classical" bit error.
- Given by NOT gate $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Phase-flip error:

- Inverts the relative phase of $|0\rangle$ and $|1\rangle$. Has no classical analogue!
- Given by the matrix $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Combination:

- Combining bit-flip and phase-flip gives $Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = XZ$.

Error Basis

Pauli Matrices

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- vector space basis of all 2×2 matrices
- unitary matrices which generate a *finite* group

Error Basis for many Qubits/Qudits

$\mathcal{E}$ error basis for subsystem of dimension d with $I \in \mathcal{E}$

$\Longrightarrow \mathcal{E}^{\otimes n}$ error basis with elements

$$E := E_1 \otimes \ldots \otimes E_n, \quad E_i \in \mathcal{E}$$

weight of E: number of factors $E_i \neq I$

Quantum Depolarizing Channel

- analogue of the binary symmetric channel (BSC) and uniform symmetric channel (USC)
- either transmits a quantum state faithfully or replaces it with a completely random (maximally mixed) state

$$\rho \mapsto (1-p)\rho + p\mathbb{1} = (1-p)\rho + \frac{p}{d^2}\sum_{E\in\mathcal{E}} E\rho E^\dagger,$$

  where $\mathcal{E}$ is (nice) unitary error basis

- after *discretisation*, each error operator E different from identity is equally likely

other quantum channels:
dephasing channel, amplitude damping channel, asymmetric channels

warning: A quantum channel implies a discrete channel, not the other way round.

Quantum Error-Correcting Codes (QECC)

- **subspace** $\mathcal{C}$ of a complex vector space $\mathcal{H} \cong \mathbb{C}^N$
  usually: $\mathcal{H} \cong \mathbb{C}^q \otimes \mathbb{C}^q \otimes \ldots \otimes \mathbb{C}^q =: (\mathbb{C}^q)^{\otimes n}$ "n qudits"

- **errors:** described by linear transformations acting non-trivially on some of the subsystems (local errors)

- **notation:** $\boxed{\mathcal{C} = ((n, K, d))_q}$ or $\boxed{\mathcal{C} = [\![n, k, d]\!]_q}$
  K-dimensional or q^k-dimensional subspace $\mathcal{C}$ of $(\mathbb{C}^q)^{\otimes n} \cong \mathbb{C}^{q^n}$

- **minimum distance** d:
  - detection of all errors acting nontrivially on $d-1$ subsystems
  - correction of all errors acting on $\lfloor (d-1)/2 \rfloor$ subsystems
  - correction of all erasures affecting up to $d-1$ subsystems
    [Grassl, Beth, & Pellizzari, *Codes for the Quantum Erasure Channel*, PRA **56**, pp. 33–38 (1997)]

Bit-flips and Phase-flips

Let $C \leq \mathbb{F}_2^n$ be a linear code. Then the image of the state

$$\frac{1}{\sqrt{|C|}} \sum_{\boldsymbol{c} \in C} |\boldsymbol{c}\rangle$$

under first a phase-flip $\boldsymbol{z} \in \mathbb{F}_2^n$ and then a bit-flip $\boldsymbol{x} \in \mathbb{F}_2^n$ is given by

$$\frac{1}{\sqrt{|C|}} \sum_{\boldsymbol{c} \in C} (-1)^{\boldsymbol{z} \cdot \boldsymbol{c}} |\boldsymbol{c} + \boldsymbol{x}\rangle.$$

Hadamard transform $H \otimes \ldots \otimes H$ maps this to

$$\frac{(-1)^{\boldsymbol{x} \cdot \boldsymbol{z}}}{\sqrt{|C^\perp|}} \sum_{\boldsymbol{b} \in C^\perp} (-1)^{\boldsymbol{x} \cdot \boldsymbol{b}} |\boldsymbol{b} + \boldsymbol{z}\rangle$$

CSS Codes

Introduced by R. Calderbank, P. Shor, and A. Steane
[Calderbank & Shor PRA, **54**, 1098–1105, 1996]
[Steane, PRL **77**, 793–797, 1996]

Construction: Let $C_1 = [n, k_1, d_1]_q$ and $C_2 = [n, k_2, d_2]_q$ be classical linear codes with $C_2^\perp \le C_1$. Let $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K\}$ be representatives for the cosets $C_1/C_2^\perp$. Define quantum states

$$|\boldsymbol{x}_i + C_2^\perp\rangle := \frac{1}{\sqrt{|C_2^\perp|}} \sum_{\boldsymbol{y} \in C_2^\perp} |\boldsymbol{x}_i + \boldsymbol{y}\rangle$$

Theorem: Then the vector space $\mathcal{C}$ spanned by these states is a quantum code with parameters $[\![n, k_1 + k_2 - n, d]\!]_q$ where

$$d \ge \min\left\{\operatorname{wgt}(C_1 \setminus C_2^\perp), \operatorname{wgt}(C_2 \setminus C_1^\perp)\right\} \ge \min(d_1, d_2).$$

($d = \min(d_1, d_2)$ when $C_2^\perp = C_1$)

CSS Codes — how they work

Basis states:

$$|\boldsymbol{x}_i + C_2^\perp\rangle = \frac{1}{\sqrt{|C_2^\perp|}} \sum_{\boldsymbol{y}\in C_2^\perp} |\boldsymbol{x}_i + \boldsymbol{y}\rangle$$

Suppose a bit-flip error $\boldsymbol{b}$ happens to $|\boldsymbol{x}_i + C_2^\perp\rangle$:

$$\frac{1}{\sqrt{|C_2^\perp|}} \sum_{\boldsymbol{y}\in C_2^\perp} |\boldsymbol{x}_i + \boldsymbol{y} + \boldsymbol{b}\rangle$$

Now, we introduce an ancilla register initialized in $|0\rangle$ and compute the syndrome.

CSS Codes — how they work

Let H_1 be the parity check matrix of C_1, *i. e.*, $\boldsymbol{x}H_1^t = 0$ for all $\boldsymbol{x} \in C_1$.

$$\frac{1}{\sqrt{|C_2^\perp|}} \sum_{\boldsymbol{y} \in C_2^\perp} |\boldsymbol{x}_i + \boldsymbol{y} + \boldsymbol{b}\rangle |(\boldsymbol{x}_i + \boldsymbol{y} + \boldsymbol{b})H_1^t\rangle = \frac{1}{\sqrt{|C_2^\perp|}} \sum_{\boldsymbol{y} \in C_2^\perp} |\boldsymbol{x}_i + \boldsymbol{y} + \boldsymbol{b}\rangle |\boldsymbol{b}H_1^t\rangle$$

measure the ancilla to obtain $\boldsymbol{s} = \boldsymbol{b}H_1^t$

use this to correct the error by a conditional operation which flips the bits in $\boldsymbol{b}$

Phase-flips: Suppose we have the state

$$\frac{1}{\sqrt{|C_2^\perp|}} \sum_{\boldsymbol{y} \in C_2^\perp} (-1)^{(\boldsymbol{x}_i + \boldsymbol{y}) \cdot \boldsymbol{z}} |\boldsymbol{x}_i + \boldsymbol{y}\rangle$$

$H^{\otimes n}$ yields a superposition over a coset of C_2 which has a bit-flip.

Correct it as before (with a parity check matrix for C_2).

independent correction of bit-/phase flips $\Longrightarrow$ correction of combinations

Quantum Stabilizer Codes

[Gottesman, PRA **54** (1996); Calderbank, Rains, Shor, & Sloane, IEEE-TIT **44** (1998)]

Basic Idea

Decomposition of the complex vector space into eigenspaces of operators.

Error Basis for Qudits

[A. Ashikhmin & E. Knill, Nonbinary quantum stabilizer codes, IEEE-TIT **47** (2001)]

$$\mathcal{E} = \{X^\alpha Z^\beta : \alpha, \beta \in \mathbb{F}_q\},$$

where (you may think of $\mathbb{C}^q \cong \mathbb{C}[\mathbb{F}_q]$)

$$X^\alpha = \sum_{x\in\mathbb{F}_q} |x+\alpha\rangle\langle x| \qquad \text{for } \alpha \in \mathbb{F}_q$$

$$\text{and} \quad Z^\beta = \sum_{z\in\mathbb{F}_q} \omega^{\mathrm{Tr}(\beta z)}|z\rangle\langle z| \quad \text{for } \beta \in \mathbb{F}_q \ (\omega = \omega_p = \exp(2\pi i/p))$$

Stabilizer Codes

common eigenspace of an Abelian subgroup $\mathcal{S}$ of the group $\mathcal{G}_n$ with elements

$$\omega^\gamma(X^{\alpha_1}Z^{\beta_1})\otimes(X^{\alpha_2}Z^{\beta_2})\otimes\ldots\otimes(X^{\alpha_n}Z^{\beta_n}) =: \omega^\gamma X^{\boldsymbol{\alpha}}Z^{\boldsymbol{\beta}},$$

where $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{F}_q^n$, $\gamma\in\mathbb{F}_p$

quotient group:

$$\overline{\mathcal{G}}_n := \mathcal{G}_n/\langle\omega I\rangle \cong (\mathbb{F}_q\times\mathbb{F}_q)^n \cong \mathbb{F}_q^n\times\mathbb{F}_q^n \quad \textit{as additive spaces}$$

$\mathcal{S}$ Abelian subgroup

$\iff (\boldsymbol{\alpha},\boldsymbol{\beta})\star(\boldsymbol{\alpha}',\boldsymbol{\beta}')=0$ for all $\omega^\gamma(X^{\boldsymbol{\alpha}}Z^{\boldsymbol{\beta}})$, $\omega^{\gamma'}(X^{\boldsymbol{\alpha}'}Z^{\boldsymbol{\beta}'})\in\mathcal{S}$,

where $\star$ is a symplectic inner product on $\mathbb{F}_q^n\times\mathbb{F}_q^n$

Stabilizer codes correspond to trace symplectic self-orthogonal codes over $\mathbb{F}_q^n\times\mathbb{F}_q^n$

Trace Symplectic Self-Orthogonal Codes

most general:

additive codes $C \subset \mathbb{F}_q^n \times \mathbb{F}_q^n$ that are self-orthogonal with respect to

$$(\boldsymbol{v}, \boldsymbol{w}) \star (\boldsymbol{v}', \boldsymbol{w}') := \operatorname{Tr}(\boldsymbol{v} \cdot \boldsymbol{w}' - \boldsymbol{v}' \cdot \boldsymbol{w}) = \operatorname{Tr}\Big(\sum_{i=1}^{n} v_i w_i' - v_i' w_i\Big) \tag{1}$$

special cases:

for $\mathbb{F}_q$-linear codes $C \subset \mathbb{F}_q^n \times \mathbb{F}_q^n$, (1) reduces to the symplectic inner product

$$(\boldsymbol{v}, \boldsymbol{w}) \star (\boldsymbol{v}', \boldsymbol{w}') := \boldsymbol{v} \cdot \boldsymbol{w}' - \boldsymbol{v}' \cdot \boldsymbol{w} = \sum_{i=1}^{n} v_i w_i' - v_i' w_i$$

for $\mathbb{F}_{q^2}$-linear codes $C \subset \mathbb{F}_{q^2}^n$ (1) reduces to the Hermitian inner product

$$\boldsymbol{x} \star \boldsymbol{y} := \sum_{i=1}^{n} x_i^q y_i \qquad \text{(identifying } \mathbb{F}_q \times \mathbb{F}_q \text{ and } \mathbb{F}_{q^2}\text{)}$$

Quantum Codes from Classical Codes

Hermitian self-orthogonal code

linear code $C = [n, k, d']_{q^2} \leq \mathbb{F}_{q^2}^n$ that is self-orthogonal with respect to the Hermitian inner product

$$\boldsymbol{x} \star \boldsymbol{y} := \sum_{i=1}^{n} x_i^q y_i,$$

i. e., $C \leq C^\star = \{\boldsymbol{x} \in \mathbb{F}_{q^2}^n \mid \forall \boldsymbol{y} \in C \colon \boldsymbol{x} \star \boldsymbol{y} = 0\}$

Theorem: (Hermitian construction)

Let $C = [n, k, d']_{q^2}$ be a Hermitian self-orthogonal code and let

$$d := \min\{\mathrm{wgt}(\boldsymbol{c}) \colon \boldsymbol{c} \in C^\star \setminus C\} \geq d_{\mathsf{min}}(C^\star).$$

Then there exists a quantum code $\mathcal{C} = [\![n, n-2k, d]\!]_q$. $\mathcal{C}$ is *pure* iff $d = d_{\mathsf{min}}(C^\star)$.

[Ketkar et al., *Nonbinary stabilizer codes over finite fields*, IEEE-TIT **52**, pp. 4892–4914 (2006)]

A Connection to Boolean Functions

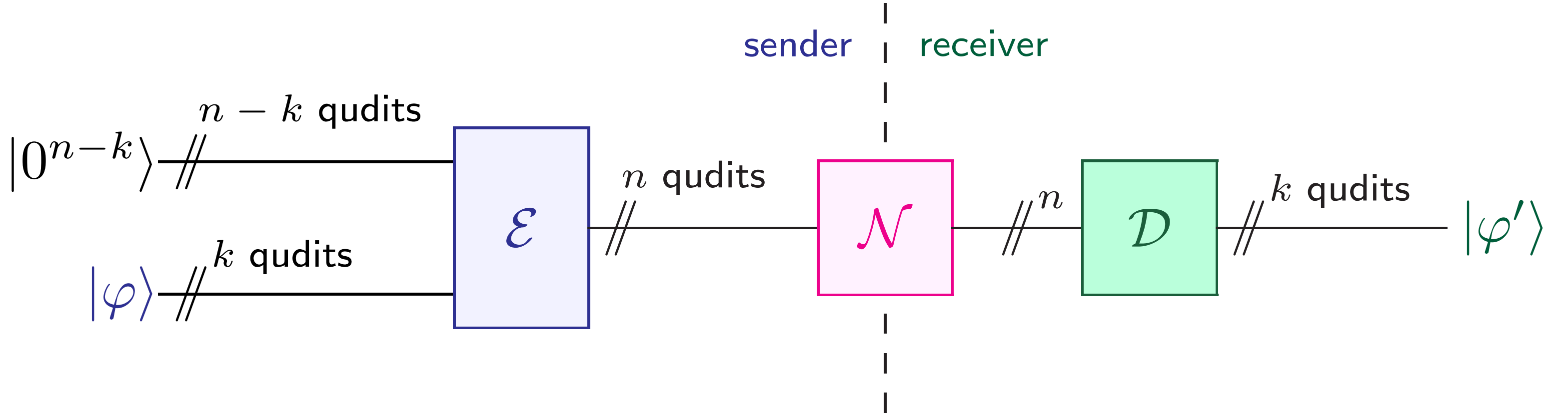
[Aggarwal & Calderbank, IEEE Trans. Inf. Theory, **54**, 1700–1707 (2008)]

- $f(x_1,\ldots,x_n) = \sum_{i=0}^{2^n-1} y_i x_1^{c_1}\cdots x_n^{c_n}$ a Boolean function
- $P_1,\ldots,P_n$ distinct, mutually commuting orthogonal projection operators on $\mathbb{C}^{2^n}$ of rank 2^{n-1}
- $P_f = f(P_1,\ldots,P_n) = \bigvee_{i=0}^{2^n-1} y_i P_1^{c_1}\cdots P_n^{c_n}$
  is a projection operator, where $P\vee P' := P+P'-PP'$ and $P^0 := I-P$
- $\operatorname{rank}(P_f) = |\{(x_1,\ldots,x_n)\in\mathbb{F}_2^n\colon f(x_1,\ldots,x_n)=1\}| = \operatorname{wgt}(f) = K$
- conditions on the *complementary set* of f

$$\mathsf{Cset}_f = \Big\{\boldsymbol{a}\colon \sum_{\boldsymbol{x}\in\mathbb{F}_2^n} f(\boldsymbol{x})f(\boldsymbol{x}+\boldsymbol{a}) = 0\Big\}$$

imply that P_f projects onto a QECC $((n,K,d))_2$

Quantum Error-Correcting Code (QECC)



Scheme of a communication protocol using a QECC $[\![n, k, d]\!]_q$

quantum Singleton bound: $\boxed{2d \leq n + 2 - k}$

[E. Rains, Nonbinary Quantum Codes, IEEE-TIT **45**, pp. 1827–1832 (1999)]

open question:

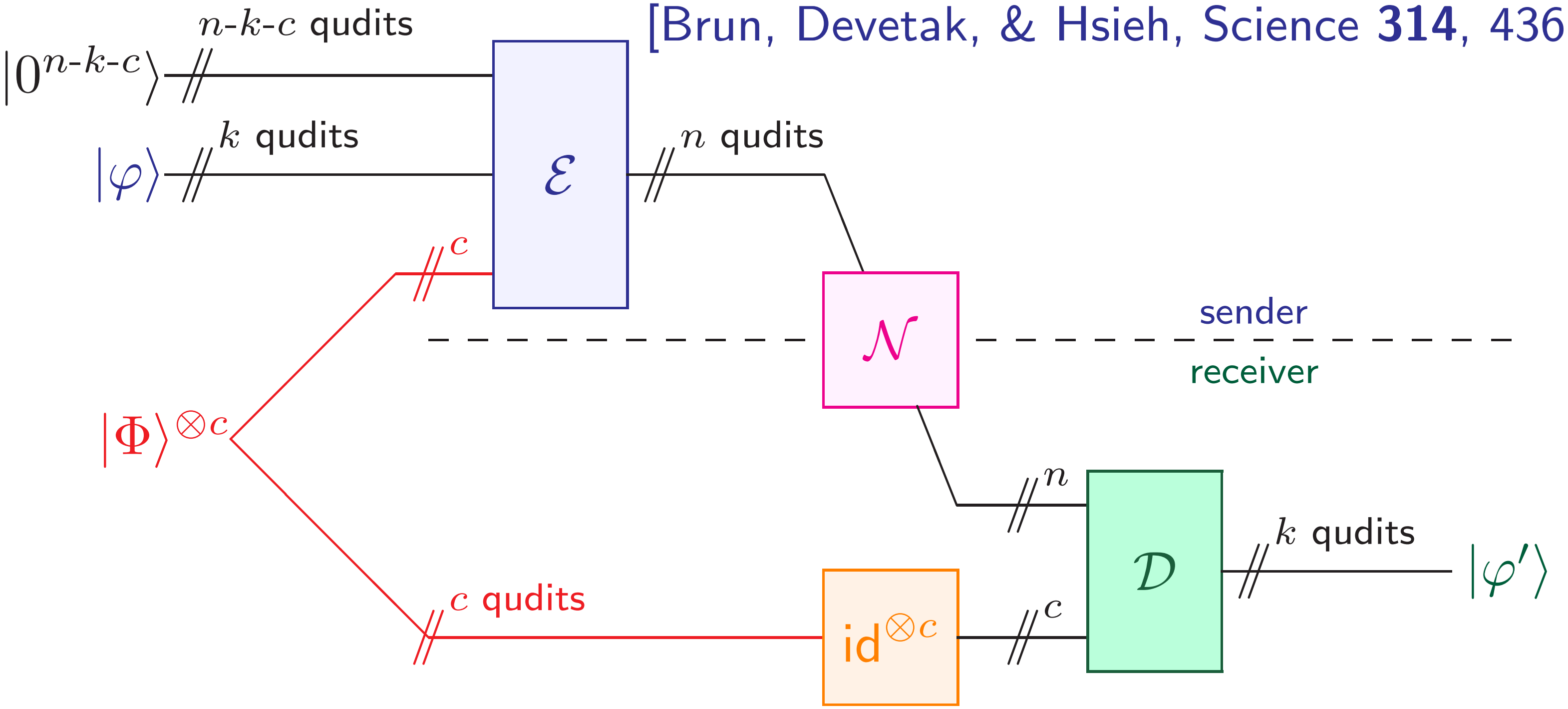
For which n, k do quantum MDS codes exist?

QMDS conjecture:

$n \leq q^2 + 1$ for $d > 2$ ($n \leq q^2 + 2$ in some cases)

Entanglement-Assisted QECC

[Brun, Devetak, & Hsieh, Science **314**, 436 (2006)]



Communication scheme using an entanglement-assisted QECC $\mathcal{C} = [\![n, k, d; c]\!]_q$

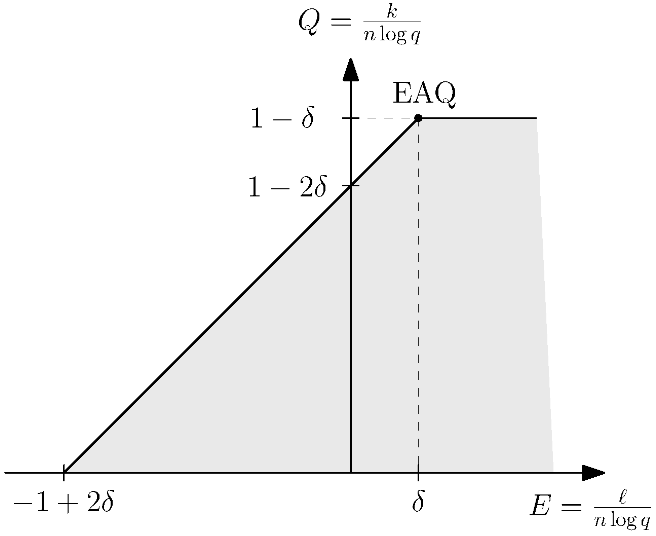
similar to a QECC of length $n + c$

$\Longrightarrow$ **conjectured bound:** $\boxed{2d \leq n + c + 2 - k}$

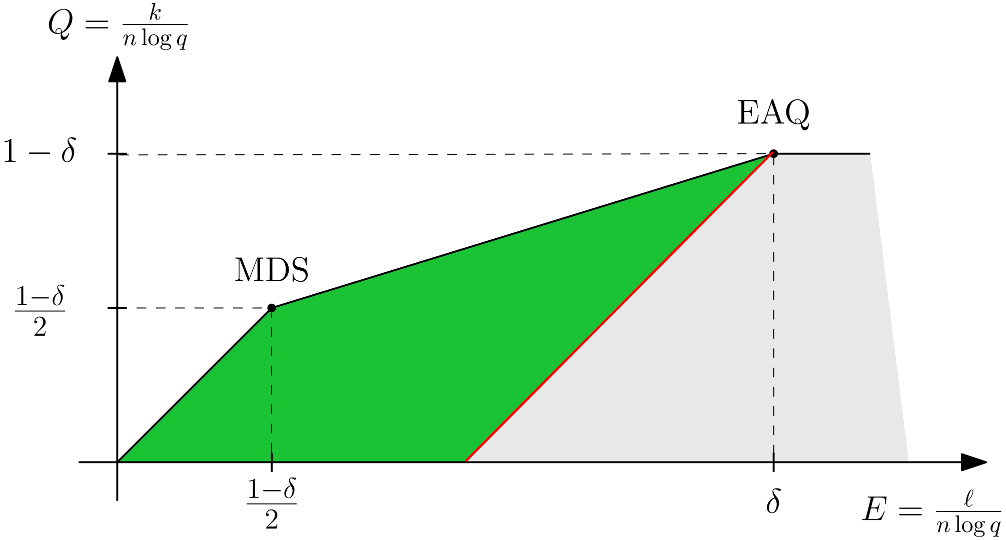
[Brun, Devetak, & Hsieh, arXiv:quant-ph/0608027]

Singleton-type Bounds

[Grassl, Huber, Winter, IEEE-TIT **68**, pp. 3942–3950 (2022)]



$(a) \quad \delta = \frac{d-1}{n} < \frac{1}{2}$



$(c) \quad \delta > \frac{1}{2}$

$\mathcal{C} = [\![n, k, d; c]\!]_q$:

(a) $\quad d < n/2 + 1 : \quad k \le \min\{n + c + 2 - 2d, n - d + 1\}$

(c) $\quad d \ge n/2 + 1 : \quad k \le c,$

$$k \le \frac{n-d+1}{3d-3-n}\big(c + (2d - 2 - n)\big),$$

$$k \le n - d + 1$$

Constructing Entanglement-Assisted QECC

Hermitian hull: $\mathrm{Hull}(C) = C \cap C^\star$

C is Hermitian self-orthogonal $\iff \mathrm{Hull}(C) = C$

Theorem: (Hermitian construction)
Let $C = [n, k, d']_{q^2}$ be a linear code with $c = k - \dim \mathrm{Hull}(C)$ and let

$$d := \min\{\mathrm{wgt}(\boldsymbol{c})\colon \boldsymbol{c} \in C^\star \setminus \mathrm{Hull}(C)\} \geq d_{\textsf{min}}(C^\star).$$

Then there exists an entanglement-assisted quantum code $\mathcal{C} = [\![n, n-2k+c, d; c]\!]_q$.

[Galindo et al., *Entanglement-assisted quantum error-correcting codes over arbitrary finite fields*, Quantum Inform. Proc., (2019)]

Proof idea: Lengthen the code C by c positions to make it self-orthogonal.
Additional positions correspond to c maximally entangled states.

LCD Codes

C is a linear complementary dual (LCD) code $\Longleftrightarrow \mathrm{Hull}(C) = C \cap C^\perp = \{\mathbf{0}\}$

Theorem [Carlet et al., IEEE-TIT **64** pp. 3010–3017 (2018)]
For $q > 2$, any linear code over $\mathbb{F}_{q^2}$ is monomially equivalent to a Hermitian LCD code.

Corollary
Given a linear code $C = [n, k, d]_{q^2}$, there exists an EAQECC $\mathcal{C} = [\![n, k, d; n-k]\!]_q$.

This is the point EAQ in the previous diagram.

but:
EAQECC from LCD codes require the maximal amount $c = n - k$ of entanglement.

Decreasing the Hull Dimension

[Luo, Ezerman, Grassl, Ling, arXiv:2207.05647]

Lemma

$$\dim \mathrm{Hull}(C) = k - \mathrm{rank}(GG^\star) \text{ and } c = \mathrm{rank}(GG^\star)$$

where G is a generator matrix of C and $G^\star$ its conjugate transpose

Theorem

For $C = [n, k, d]_{q^2}$ with $q > 2$ and $\dim \mathrm{Hull}(C) = \ell$, there exist equivalent codes C' with $\dim \mathrm{Hull}(C') = \ell'$ for each $\ell' \in \{0, 1, \ldots, \ell\}$.

Proof idea: Decompose C into the hull and its complement. Multiply the information set of the hull by some a_i with $a_i^{q+1} \neq 1$ to reduce the dimension of the hull.

Propagation Rule

Hermitian construction of a pure EAQECC $[\![n, k, d; c]\!]_q$ with $q > 2$ and $\dim \mathrm{Hull}(C) = \ell$ implies codes $[\![n, k + i, d; c + i]\!]_q$ for $i \in \{1, \ldots, \ell\}$.

Increasing the Hull Dimension

Theorem

The maximal dimension of the Hermitian hull is $\ell_{\text{max}} = \dim C - c_{\text{min}}$, where

$$c_{\text{min}} = \min\left\{\operatorname{rank}\left(G \operatorname{diag}(b_1, \ldots, b_n)\, G^\star\right) : b_i \in \mathbb{F}_q^*\right\}.$$

Theorem [E. Rains, IEEE-TIT **45**, pp. 1827–1832 (1999)]

A code $C = [n, k, d]_{q^2}$ is equivalent to a Hermitian self-orthogonal code iff the puncture code

$$P(C) = \left(C \star C\right)^\perp = \{(c_1\tilde{c}_1^q, \ldots, c_n\tilde{c}_n^q) : \boldsymbol{c}, \tilde{\boldsymbol{c}} \in C\}^\perp$$

contains a word $\boldsymbol{b} \in \mathbb{F}_q^n$ of weight n.

Open problem:

How to efficiently compute c_{min} and find $\boldsymbol{b}$?

More Propagation Rules

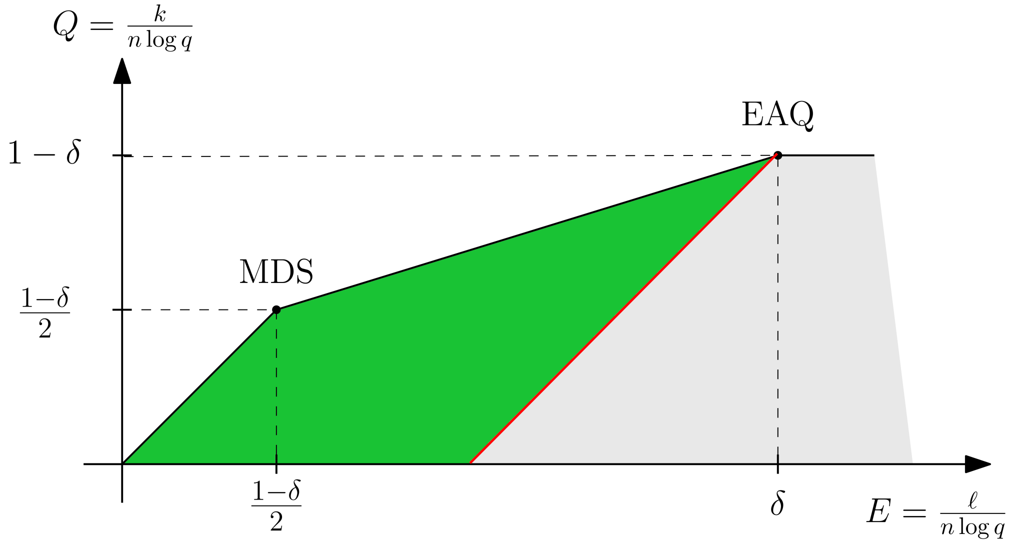
non-trivial propagation rules from [Luo, Ezerman, Grassl, Ling, arXiv:2207.05647]

(6) increasing the dimension of a pure quantum code with $q > 2$ by using extra entanglement, provided that $c \leq n - k - 2$:
$[\![n, k, d; c]\!]_q \longrightarrow [\![n, k+1, d; c+1]\!]_q$

(7) reducing the length by using extra entanglement, provided that $c \leq n - k - 2$:
$[\![n, k, d; c]\!]_q \longrightarrow [\![n-1, k, d; c+1]\!]_q$

(8) shortening a pure quantum code:
$[\![n, k, d; c]\!]_q \longrightarrow [\![n-1, k+1, d-1; c]\!]_q$

Singleton-type Bounds [arXiv:2207.05647]



$(c) \quad \delta > \frac{1}{2}$

Theorem

EAQECC $[\![n, k, d; c]\!]_q$ from the Hermitian or CSS-like construction obey the bound

$$k \leq \min\{n + c + 2 - 2d, n - d + 1\}$$

using classical MDS codes and teleportation, we achieve the point MDS

[M. Grassl, Phys. Rev. A **103** (2021)]

Summary & Outlook

- trace-symplectic self-orthogonal codes yield QECC
- any code $[n, k, d]_{q^2}$ can be used to construct EAQECC
- random codes have a small (Hermitian) hull, i.e., require a lot of entanglement
- goal: find codes with a large Hermitian hull, i.e., requiring little entanglement
- propagation rules allow to utilise more entanglement
  arXiv:2207.05647
- initial online tables for qubits and qutrits at
  `http://eaqecc.codetables.de`
- find new constructions beating the bound $k \leq n + c + 2 - 2d$ for $d > n/2$

Thank you!

Danke! Merci!

Hvala!

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