New Results on Complementary Sequence Sets

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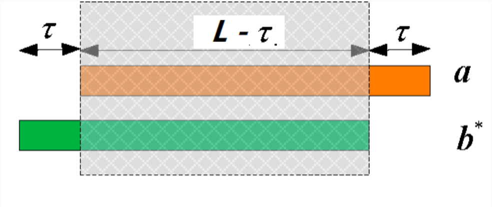
Outline

Complementary Sequences : Background

Suppose two length-L sequences $\mathbf{a} = \{a(t)\}$ and $\mathbf{b} = \{b(t)\}$. The **aperiodic correlation function** of $\mathbf{a}$ and $\mathbf{b}$ for time-shift τ is defined as



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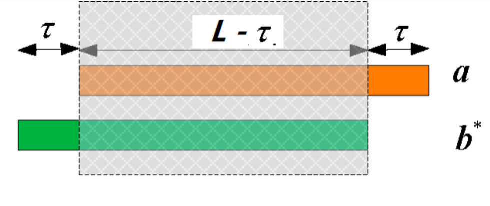

$$\rho_{\mathbf{a},\mathbf{b}}(\tau) = \sum_{t=0}^{L-1} a(t)b(t+\tau)^*$$

where $a(t), b(t) = 0$ for $t \notin \{0, 1, \ldots, L-1\}$

Auto-correlation: $\mathbf{a} = \mathbf{b}$

Perfect Complementary Sequences

A sequence is called perfect aperiodic/periodic complementary sequence if it's correlation function is zero for all non-zero time shifts.

- There is no binary sequence **a**, where $\rho_{\mathbf{a}}(\tau) = 0 \ \forall \ \tau \neq 0$.
- Although there exist a perfect binary sequence, $\mathbf{a} = (0001)$, with respect to periodic correlation function, i.e.,

$$\rho_{\mathbf{a}}(\tau) + \rho^*(\mathbf{a})(L - \tau) = 0 \ \forall \ \tau \neq 0.$$

Golay Complementary Pair (GCP)

A pair of sequences $(\mathbf{a}, \mathbf{b})$ is said to be a GCP if

$$\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) = 0, \ 0 < |\tau| < L.$$

$\mathbf{a} = (0000001101100101)$ and $\mathbf{b} = (0101011000110000)$

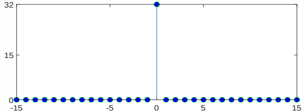

Figure 1: Auto-correlation Sum Plot of **a** and **b**

τ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\rho_{\mathbf{a}}(\tau)$	16	1	0	5	0	-5	0	-1	0	1	0	1	0	-1	0	-1
$\rho_{\mathbf{b}}(\tau)$	16	-1	0	-5	0	5	0	1	0	-1	0	-1	0	1	0	1
$\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)$	32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Complementary Set (CS)

A set containing more than two length-L sequences with aperiodic auto-correlation sum equating zero is called a CS.

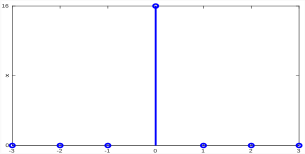

Figure 2: Plot of the sum of AACFs of $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Let $\mathbf{a}_0 = [0010]$, $\mathbf{a}_1 = [0011]$, $\mathbf{a}_2 = [0110]$, and $\mathbf{a}_3 = [0000]$.

τ	-3	-2	-1	0	1	2	3
$A(\mathbf{a}_0)(\tau)$	-1	2	-3	4	-3	2	-1
$A(\mathbf{a}_1)(\tau)$	-1	-2	1	4	1	-2	-1
$A(\mathbf{a}_2)(\tau)$	1	-2	-1	4	-1	-2	1
$A(\mathbf{a}_3)(\tau)$	1	2	3	4	3	2	1
$\sum_{i=0}^{3} A(\mathbf{a}_i)(\tau)$	0	0	0	16	0	0	0

Historical Background of Complementary Sequences

- GCPs were first introduced by Marcel Golay in 1949[1]
- Golay used them in infrared multislit spectrometry.
- Later they have been applied to OFDM.
- Binary Golay Sequences are known to exist for the lengths $2^a 10^b 26^c$ [2].
- The idea of GCPs were extended to CSs by Tseng and Liu [3].

[1] M. J. E. Golay, Multislit spectroscopy,Journal of the Optical Society of America, vol. 39, pp. 437-444, 1949.

[2] R. Turyn," Hadamard matrices, Baumert-Hall units, four-symbol sequences, pulse compression, and surface wave encodings", in Journal of Combinatorial Theory, Series A,vol. 16, pp. 313–333, 1974.

[3] C.C. Tseng and C. Liu, "Complementary sets of sequences," IEEE Transactions on Information Theory, vol. IT-18, no. 5, pp. 644–652, Sep. 1972.

Complementary Sequences in OFDM System

PAPR Control with Code/Sequences Design

- $\mathbf{a} = \{a(t)\}_{t=0}^{L-1}$ over $\mathbb{Z}_q$ is modulated: $a(t) \to \xi_q^{a(t)},\ \xi_q = e^{(2\pi\sqrt{-1}/q)}$
- the corresponding transmitted signal is the real part of

$$s(t) = \sum_{i=0}^{L-1} \xi_q^{a(i)+qf_i t},$$

where f_i is the frequency of the i-th carrier, with envelop power $P_{\mathbf{a}}(t) = |s(t)|^2$ given by

$$P_{\mathbf{a}}(t) = L + 2\Re\left(\sum_{\tau=1}^{L-1} \rho_{\mathbf{a}}(\tau)\xi_q^{-q\tau\Delta ft}\right),\ 0 \le \Delta ft \le 1$$

- the PAPR of the signal or a sequence $\mathbf{a}$ is given by

$$\mathrm{PAPR}(\mathbf{a}) := 1 + \frac{2 \sup_{0\le\theta\le1} \Re\left(\sum_{\tau=1}^{L-1} \rho_{\mathbf{a}}(\tau)\xi^{-\tau\theta}\right)}{L}$$

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- given M sequences $\mathbf{a}_i$'s, their PAPR sum is given by

$$\sum_i \mathrm{PAPR}(\mathbf{a}_i) = M + \frac{2 \sup_{0\le\theta\le 1} \Re\left(\sum_{\tau=1}^{L-1}\left(\sum_{i=1}^{M-1}\rho_{\mathbf{a}_i}(\tau)\right)\xi^{-\tau\theta}\right)}{L} \tag{1}$$

Therefore, if

$$\sum_i \rho_{\mathbf{a}_i}(\tau) = 0 \quad \forall\, \tau \neq 0,$$

then

$$\mathrm{PAPR}(\mathbf{a}_i) \le M \quad \forall\, i = 1, \ldots, M$$

Complementary Sequences Sets in MC-CDMA Systems

Multicarrier systems: single sequences set (SS) extended to multiple SSs

$$\mathcal{C} = \begin{bmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_M \end{bmatrix} \Longrightarrow \mathcal{C}^{(k)} = \begin{bmatrix} \mathbf{c}_1^{(k)} \\ \vdots \\ \mathbf{c}_M^{(k)} \end{bmatrix}, 1 \leq k \leq K$$

- For $1 \leq k_1, k_2 \leq K$, $\rho_{\mathcal{C}^{k_1},\mathcal{C}^{k_2}}(\tau) = \sum_{j=1}^{M} \rho_{\mathbf{c}_j^{k_1},\mathbf{c}_j^{k_2}}(\tau)$.
- $\theta_{\text{auto}} = \max\{|\rho_{\mathcal{C}^k}(\tau)| : k = 1, \ldots, K,\ 0 < |\tau| < L\}$, $\theta_{\text{cross}} = \max\{|\rho_{\mathcal{C}^{k_1},\mathcal{C}^{k_2}}(\tau)| : 1 \leq k_1 \neq k_2 \leq K, 0 \leq |\tau| < L\}$, and $\theta = \max\{\theta_{\text{auto}}, \theta_{\text{cross}}\}$.
- When $\theta = 0$, and $K = M$, $\mathcal{C}$ is called complete complementary codes (CCCs) [Suehiro-Hatori 1998, Rathinakumar and Chaturvedi 2008, Liu-Guan-Parampalli 2014]
- entire period to partial zone: ZCZ-CCCs ($K = M\lfloor L/Z \rfloor$) [Sarkar-Majhi-Liu 2019]
- entire period with $\theta \neq 0$: quasi-complementary sequence sets (QCSSs)

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Main Approaches

- generalized Boolean functions (GBFs)
  - many important contributions from Fan, Liu, Majhi, Sarkar,Yang, Zhou, ...
- paraunitary (PU) matrices [Das-Budisin-Majhi-Liu-Guan 2018, Wang-Ma-Gong-Xue 2021[1]]

[1]Many known GBF-based constructions of CSSs and CCCs literature can be explained from this approach

Generalized Boolean functions and Graph

A quadratic q-ary function from $\mathbb{Z}_p^m$ to $\mathbb{Z}_q$, where $p \mid q$, can be expressed as

$$f = \sum_{0 \le i,j < m} q_{i,j} x_i x_j + \sum_{0 \le j < m} c_j x_j + c$$

Let each variable x_i be a vertex, and label an edge between two vertices x_i, x_j if $q_{i,j} \neq 0$.

Figure 3: Graph of the function $2x_0x_2 + 2x_2x_1 + x_2 + 1$

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Generalized Boolean functions and Sequences

For a q-ary function $f : \mathbb{Z}_p^m \rightarrow \mathbb{Z}_q$, we define the corresponding sequence as

$$\psi(f) = (\xi_q^{f_0}, \xi_q^{f_1}, \ldots, \xi_q^{f_{p^m-1}}),$$

where $f_i = f(i_0, i_1, \ldots, i_{m-1})$, $(i_0, i_1, \ldots, i_{m-1}) \in \mathbb{Z}_p^m$ is the p-ary vector representation of $i \in \mathbb{Z}_{p^m}$, and $\xi_q = e^{2\pi\sqrt{-1}/q}$

GBF Based Construction of Complementary Sequences

- Generalized Boolean functions: $\mathbb{Z}_p^m \to \mathbb{Z}_q$
- GCPs can be extended in terms of generalized Boolean functions from $\mathbb{Z}_2^m$ to $\mathbb{Z}_{2^h}$ [Davis-Jedwab 1999]:

$$(\psi(f), \psi(f + 2^{h-1}x_{\pi(1)} + c)$$

where f is given by the function

$$f(x_0, \ldots, x_{m-1}) = 2^{h-1}\sum_{k=1}^{m-1} x_{\pi(k)}x_{\pi(k+1)} + \sum_{k=1}^{m} c_k x_k + c$$

- Paterson later extended the construction with parameters from 2^h to even q. Moreover, he introduced an important method to **restriction**, which allows us to study **higher-degree functions** in terms of special quadratic functions. [Paterson 2000]
- Paterson's idea has been largely adopted to construct CCCs [Rathinakumar and Chaturvedi 2008].

Quasi-complementary sequences sets

It is desirable to construct QCSSs with

- large set size K
- low correlation magnitude θ
- flexible choices of M and sequence lengths L
- small alphabet q

New Bounds and Construction of QCSSs

Bounds on QCSSs

Consider a QCSS with parameters (K, M, L) where

- K: the number of sequence sets,
- M: the number of sequences in each set
- L: the sequence length

What would be its largest aperiodic correlation (sum) magnitude?

$$\theta = \max\{\theta_{\text{auto}}, \theta_{\text{cross}}\}$$

- A classic bound $K \geq M$ [Welch, 1976]

$$\theta \geq ML\sqrt{\frac{\frac{K}{M}-1}{K(2L-1)-1}}$$

- An improved bound for $\frac{K}{M} \geq 3$ [Liu-Guan-Mow, 2013]

$$\theta \geq \sqrt{ML\left(1-2\sqrt{\frac{M}{3K}}\right)}$$

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CCC-based QCSSs

Recently several constructions of QCSSs are a collection of CCCs for increasing set size K.

Informally we call them CCC-based QCSSs

Research Questions

- Can we derive **better bounds** on correlation magnitude for such CCC-based QCSSs?
  - cross-correlation sum for each CCC is zero
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A Powerful Method

- introduced by Levenshtein 1999 for $M = 1$
- tailored by Liu-Guan-Mow in 2014 for QCSSs ($M \geq 2$)
- further adjusted for CCC-based QCSSs

For two (K, M, L, θ) QCSSs $\mathcal{C}$ and $\mathcal{D}$, define a function

$$F(\mathcal{C}, \mathcal{D}) = \frac{1}{|\mathcal{C}||\mathcal{D}|} \sum_{\mathbf{X}\in\mathcal{C}, \mathbf{Y}\in\mathcal{D}} \sum_{u,v=0}^{2L-2} \left|\left\langle T^{u}\left(\mathbf{X}, \mathbf{0}_{L-1}\right), T^{v}\left(\mathbf{Y}, \mathbf{0}_{L-1}\right)\right\rangle\right|^2 w_u w_v,$$

where T is a shift operator and $\mathbf{w} = (w_0, \ldots, w_{2L-2})$ satisfying

$$w_0 + w_1 + \cdots + w_{2L-2} = 0 \text{ and } w_u \geq 0.$$

Lemma (Liu-Guan-Mow 2013)

Let $\mathcal{C}$ be a (K, M, L, θ)-QCSS. Then

$$F(\mathcal{C}, \mathcal{C}) \geq \sum_{u,v=0}^{2L-2} M(L - \tau_{u,v,L}) w_u w_v,$$

where

$$0 \leq \tau_{u,v,L} = \min\{|v-u|, 2L-1-|v-u|\} \leq L-1.$$

Theorem (Main Result 1)

Let $\mathcal{C}$ be a collection of $N \geq 2$ different (M, L)-CCCs. Then the maximum correlation magnitude θ of such a $(K = NM, L, M)$ QCSS satisfies

$$\theta^2 \geq \frac{M\left(L - Q\left(\mathbf{w}, \frac{L^2}{N}\right)\right)}{1 - \frac{1}{N}},$$

where

$$Q(\mathbf{w}, a) = a \sum_{u=0}^{2L-2} w_u^2 + \sum_{u,v=0}^{2L-2} \tau_{u,v,L} w_u w_v.$$

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Choices of Weight Vectors

The vector $\mathbf{w} = (w_0, \ldots, w_{2L-2})$ should satisfy

$$\sum_{u=0}^{2L-2} w_u = 1 \text{ and } w_u \geq 0.$$

Choice 1

Define a step function weight vector as

$$w_j = \begin{cases} \frac{1}{t}, & j = 0, 1, \ldots, t-1, \\ 0, & j = t, t+1, \ldots, 2L-2, \end{cases}$$

where $1 \leq t \leq 2L - 1$

Choice 2

Define a positive-cycle-of-a-sine-wave weight vector

$$w_j = \begin{cases} \tan \frac{\pi}{2t} \sin \frac{\pi j}{t}, & j \in \{0, 1, \ldots, t-1\} \\ 0, & j \in \{t, t+1, 2L-1\} \end{cases}$$

Improved Bounds

For a QCSS as a collection of $N \geq 2$ (M, L)-CCCs, the lower bounds on θ are improved as follows:

$$\theta^2 \geq \begin{cases} \frac{ML^2}{2L-1}, & N = 2, \text{ or } N = 3,\ 2 \leq L \leq 25, \\ ML\left(1 - \frac{L^2(2\pi^2+4N-16)-N\pi^2}{16L^2(N-1)}\right), & N = 3,\ L > 25, \\ ML\left(1 - \frac{\pi\sqrt{N(2L^2-N)}-4L}{4(N-1)L}\right), & N > 3. \end{cases}$$

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Improved Bounds

For a QCSS as a collection of $N \geq 2$ (M, L)-CCCs, the lower bounds on θ are improved as follows:

$$\theta^2 \geq \begin{cases} \frac{ML^2}{2L-1}, & N = 2, \text{ or } N = 3,\, 2 \leq L \leq 25, \\ ML\left(1 - \frac{L^2(2\pi^2+4N-16)-N\pi^2}{16L^2(N-1)}\right), & N = 3,\ L > 25, \\ ML\left(1 - \frac{\pi\sqrt{N(2L^2-N)}-4L}{4(N-1)L}\right), & N > 3. \end{cases}$$

These bounds are tighter than the earlier bounds [Welch 1974, Liu-Guan-Mow 2013]

New Constructions of QCSSs

Generalized Boolean functions have been adopted to construct GCPs, CSs and CCCs

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Example

Suppose $f : \mathbb{Z}_3^5 \to \mathbb{Z}_3$, given by $f = x_0x_2 + x_2x_1 + x_3x_4x_0 + 2x_1 + x_2 + 1$. When considering the restrictions of f on x_3 and x_4, we have

- $f|_{x_3,x_4=(0,0)} = x_0x_2 + x_2x_1 + 2x_1 + x_2 + 1$
- $f|_{x_3,x_4=(1,2)} = x_0x_2 + x_2x_1 + 2x_0 + 2x_1 + x_2 + 1$

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Main Construction

Consider a q-ary function $f : \mathbb{Z}_p^m \rightarrow \mathbb{Z}_q$ such that for each $\mathbf{c} \in \mathbb{Z}_p^n$, the graph $G(f|_{\mathbf{x}_J=\mathbf{c}})$ is a Hamiltonian path over the vertices $x_{I_{\pi(i)}}$, $i = 0, 1, \ldots, m-n-1$, with edges having identical weight q/p, where $J \subset \mathbb{Z}_m$, and $\pi : \{0, 1. \ldots, m-n-1\} \rightarrow \mathbb{Z}_m \setminus J$ is an one-to-one mapping.

Let us define the following set of q-ary functions:

$$C_t^k = \left\{ f_{d,t} = f + \frac{kq}{p}\left(\mathbf{d}\cdot\mathbf{x}_J + d_n x_{I_{\pi(0)}}\right) + \frac{q}{p}\left(\mathbf{t}\cdot\mathbf{x}_J + t_n x_{I_{\pi(m-n-1)}}\right) : 0 \le d < p^{n+1} \right\}.$$

where $(\mathbf{d}, d_n) = (d_0, d_1, \ldots, d_n)$ and $(\mathbf{t}, t_n) = (t_0, t_1, \ldots, t_n)$ is the vector representation of the integer d and $t \in \mathbb{Z}_{p^{n+1}}$.

Define a code as

$$\psi(C_t^k) = \left\{ \psi(f_{d,t}) \,|\, f_{d,t} \in C_t^k \right\}$$

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Theorem

For each $1 \leq k < p$*, the set*

$$\mathcal{C}_k = \left\{ \psi(C_t^k) \,|\, t = 0, 1, \ldots, p^{n+1} - 1 \right\}.$$

forms a (p^{n+1}, p^m)*-CCC over* $\mathbb{Z}_q$

Example

- take $m = 3$, $p = 3$, and $q = 6$ and consider

$$f(x_0, x_1, x_2) = x_0x_2 + 2x_2x_1 + x_1x_0 + x_0 + 2x_1 + x_2 + 1.$$

- $f|_{x_0=0} = 2x_1x_2 + 2x_1 + x_2 + 1$, $f|_{x_0=1} = 2x_1x_2 + 3x_1 + 2x_2 + 2$, and $f|_{x_0=2} = 2x_1x_2 + 4x_1 + 3x_2 + 1$
- from this function we can obtain several CCCs

$\mathcal{C}_1: \ (9,27)-CCCs$	$\mathcal{C}_2: \ (9,27)-CCCs$
$\psi(C_0^1)=\{\psi(f+2(d_0x_0+d_1x_1)):0\leq d<9\}$	$\psi(C_0^2)=\{\psi(f+4(d_0x_0+d_1x_1)):0\leq d<9\}$
$\psi(C_1^1)=\{\psi(f+2(d_0x_0+d_1x_1)+2x_1):0\leq d<9\}$	$\psi(C_1^2)=\{\psi(f+4(d_0x_0+d_1x_1)+2x_1):0\leq d<9\}$
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$\psi(C_4^1)=\{\psi(f+2(d_0x_0+d_1x_1)+2(x_0+x_1)):0\leq d<9\}$	$\psi(C_4^2)=\{\psi(f+4(d_0x_0+d_1x_1)+2(x_0+x_1)):0\leq d<9\}$
$\psi(C_5^1)=\{\psi(f+2(d_0x_0+d_1x_1)+2(x_0+2x_1)):0\leq d<9\}$	$\psi(C_5^2)=\{\psi(f+4(d_0x_0+d_1x_1)+2(x_0+2x_1)):0\leq d<9\}$
$\psi(C_6^1)=\{\psi(f+2(d_0x_0+d_1x_1)+4x_0):0\leq d<9\}$	$\psi(C_6^2)=\{\psi(f+4(d_0x_0+d_1x_1)+4x_0):0\leq d<9\}$
$\psi(C_7^1)=\{\psi(f+2(d_0x_0+d_1x_1)+2(2x_0+x_1)):0\leq d<9\}$	$\psi(C_7^2)=\{\psi(f+4(d_0x_0+d_1x_1)+2(2x_0+x_1)):0\leq d<9\}$
$\psi(C_8^1)=\{\psi(f+2(d_0x_0+d_1x_1)+2(2x_0+2x_1)):0\leq d<9\}$	$\psi(C_8^2)=\{\psi(f+4(d_0x_0+d_1x_1)+2(2x_0+2x_1)):0\leq d<9\}$

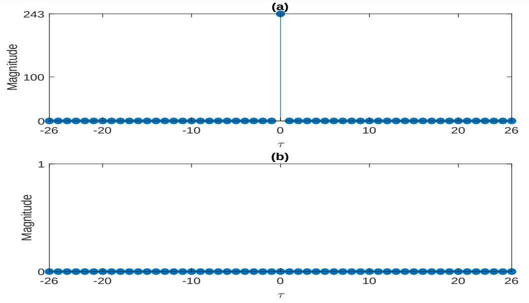

Figure 4: Correlation plot for $\mathcal{C}_k$

Theorem

Let $\mathcal{C}_1, \ldots, \mathcal{C}_{p-1}$ be as previously defined. Then for $J = \{0, \ldots, n-1\}$ and $l_{\pi(0)} = n$, the union

$$\mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_{p-1}$$

forms a (K, M, L, θ)-QCSS over $\mathbb{Z}_q$ with

- $K = p^{n+1}(p-1)$
- $M = p^{n+1}$
- $L = p^m$
- $\theta = p^m$

For sufficiently large p and $n = m - 1$, the constructed QCSSs are **asymptotically optimal** (optimal factor approaches to 1), with respect to the new bounds

Summary of Our Work

- Study a special type of QCSSs (as a union of CCCs)
  - Derive several new lower bounds on their correlation magnitude
  - Construct new asymptotically optimal QCSSs, which can have flexible set size and alphabet size

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Future Research

- How to construct optimal QCSSs w.r.t the new bounds?
- How about bounds and constructions of periodic CCCs and CCC-based QCSSs?

New Correlation Bound and Construction of Quasi-Complementary Sequence Sets

Palash Sarkar, Chunlei Li, *Senior Member, IEEE*, Sudhan Majhi, *Senior Member, IEEE*, and Zilong Liu, *Senior Member, IEEE*

***Abstract*—Quasi-complementary sequence sets (QCSSs) have attracted sustained research interests for simultaneously supporting more active users in multi-carrier code-division multiple-access (MC-CDMA) systems compared to complete complementary codes (CCCs). In this paper, we investigate a novel class of QCSSs composed of multiple CCCs. We derive a new aperiodic correlation lower bound for this type of QCSSs, which is tighter than the existing bounds for QCSSs. We then present a systematic construction of such QCSSs with a flexible alphabet size and a low maximum correlation magnitude, and also show that the constructed aperiodic QCSSs can meet the newly derived bound asymptotically.**

***Index Terms*—Multi-carrier code-division multiple-access (MC-CDMA), aperiodic correlation, complete complementary code (CCC), quasi-complementary sequence set (QCSS), multivariate function.**

I. Introduction

As a generalization of the Golay complementary pair [1], the complementary sequence set introduced by Tseng and Liu [2] consists of $M \geq 2$ constituent sequences of length L having zero aperiodic auto-correlation sum for all nonzero time shifts. A complementary sequence set is usually arranged as an $M \times L$ matrix (known as a complementary matrix or complementary code). A set of K complementary codes with the same order (M, L) is called a mutually orthogonal complementary sequence set (MOCSS) if any two distinct complementary codes have zero aperiodic cross-correlation sums for all time shifts [3]. A MOCSS has its size $K \leq M$ and it is known as a complete complementary code (CCC) when the equality is reached. Due to their ideal auto- and cross-correlation properties, CCCs have a salient feature for supporting interference-free multi-carrier code-division multiple-access (MC-CDMA) communications where users are assigned with different complementary codes from a CCC [4], [5], [6].

To support more users in MC-CDMA systems, the notion of low-correlation zone CSS, which refers to a set of (complementary) sequence sets having low maximum correlation magnitudes within a time-shift zone around the origin, was proposed [7]; in particular, when the maximum correlation magnitude within the zone is zero, it reduces to a zero-correlation zone CSS [8], [9], [10]. By extending the low correlation zone to all the non-trivial time-shifts, quasi-complementary sequence sets (QCSSs) with uniformly low maximum correlation magnitude were introduced and investigated [11]. A QCSS-based MC-CDMA system is expected to accommodate larger amount of asynchronous time-offsets, whilst supporting more users [12], [13].

A. Existing Works on the Construction and the Correlation Bound of QCSSs

In this subsection, we recall some known results on QCSSs. Let q be a positive integer and $\mathcal{A}_q = \{\xi_q^i | 0 \leq i < q\}$, where $\xi_q = \exp(2\pi\sqrt{-1}/q)$ is a q-th primitive root of unity. We denote by $\mathcal{A}_q^{M \times L}$ the set of all $M \times L$ matrices over $\mathcal{A}_q$. A subset of $\mathcal{A}_q^{M \times L}$ is termed a (K, M, L, θ)-QCSS over $\mathcal{A}_q$ if it consists of K matrices in $\mathcal{A}_q^{M \times L}$ and its maximum magnitude of aperiodic correlation sums equals a positive value θ. The multipath interference and multiuser interference in QCSS-based MC-CDMA systems are constrained by the maximum correlation sum magnitude θ, which is desired to be small. In the literature, several researchers have studied the lower bound on θ. Welch in [14] first gave the following lower bound:

Manuscript received 23 December 2022; revised 22 December 2023; accepted 26 December 2023. Date of publication 11 January 2024; date of current version 16 February 2024. The work of Palash Sarkar and Chunlei Li was supported by the Research Council of Norway under Grant 311646/O70. The work of Sudhan Majhi was supported in part by the Science and Engineering Research Board (SERB) Government of India (GoI), in part by the Core Research Grant (CRG) under Grant CRG/2022/000529, and in part by the Empowerment and Equity Opportunities for Excellence in Science (EEQ) under Grant EEQ/2022/001018. The work of Zilong Liu was supported in part by the U.K. Engineering and Physical Sciences Research Council under Grant EP/X035352/1 and Grant EP/Y000986/1, in part by the Royal Society under Grant IEC\R3\223079, and in part by the Research Council of Norway under Grant 311646/O70. *(Corresponding author: Chunlei Li.)*

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Thank You