On Cryptographic Properties of a Class of Power Permutations in Odd Characteristic

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$$f(X) = \sum_{i=0}^{q-1} a_i X^i,\ a_i \in \mathbb{F}_q.$$
- We call a polynomial $f \in \mathbb{F}_q[X]$, a permutation polynomial (PP) over $\mathbb{F}_q$ if the associated mapping $x \mapsto f(x)$ is a bijection from $\mathbb{F}_q$ to $\mathbb{F}_q$.

Differential uniformity

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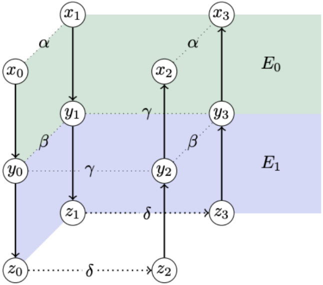

Figure: Basic Boomerang Attack

Boomerang Attacks

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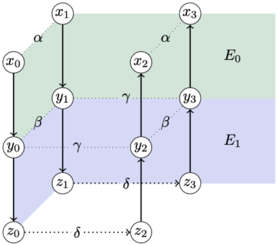
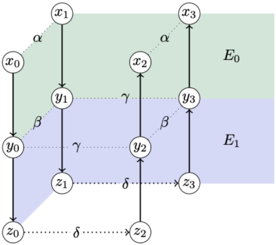
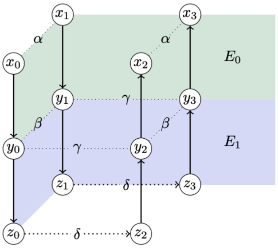
- A good differential $\gamma \xrightarrow{E_1} \beta$ over E_1 that holds with probability q.
- These two differentials can now be used to construct a distinguisher over the whole cipher.
- We start with a pair of plaintexts x_0 and x_1 with a difference α.
- When encrypting these two plaintexts, we expect the corresponding intermediate texts $y_0 := E_0(x_0)$ and $y_1 := E_0(x_1)$ to have a difference β with probability p.



- With $z_0 := E_1(y_0)$ and $z_1 := E_1(y_1)$ being the respective ciphertexts, we now construct two more ciphertexts $z_2 := z_0 + \delta$ and $z_3 := z_1 + \delta$ by adding the difference δ to each of z_0 and z_1.
- Then the pairs (z_0, z_2) and (z_1, z_3) both have a difference of δ, the ciphertext difference in the second differential.
- Decrypting these two ciphertexts, provides us with two more intermediate texts, $y_2 := E_1^{-1}(z_2)$ and $y_3 := E_1^{-1}(z_3)$ and two more plaintexts, $x_2 := E_0^{-1}(y_2)$ and $x_3 := E_0^{-1}(y_3)$.
- Assuming independence of the two ciphertext pairs (z_0, z_2) and (z_1, z_3), both of their respective intermediate pairs (y_0, y_2) and (y_1, y_3) will have a differences of γ with probability q^2.



- Combining this with the probability that (x_0, x_1) follows the first differential, we have with probability pq^2 that $y_0 + y_1 = \beta$, $y_0 + y_2 = \gamma$ and $y_1 + y_3 = \gamma$.
- This forces the difference between y_2 and y_3 to be β.
- Again assuming independence from the other pairs, the pair (y_2, y_3) will follow the first differential with probability p, resulting in a plaintext difference of α between x_2 and x_3.
- Taking all of these steps together, we estimate that the probability to see a difference α between x_2 and x_3 is equal to p^2q^2.



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- In this paper the BCT entries were defined for permutation functions in even characteristic and the knowledge of the inverse of the permutation was required to compute the BCT entries
- In 2019, Li et al. gave an equivalent technique to compute BCT, which does not require the compositional inverse of the permutation polynomial $f(X)$ at all

Boomerang Uniformity

- For any $a, b \in \mathbb{F}_q$, the BCT entry of the function f at point (a, b), denoted by $\mathcal{B}_f(a, b)$, is the number of solutions $(x, y) \in \mathbb{F}_q \times \mathbb{F}_q$ of the following system of equations

$$\begin{cases} f(X) - f(Y) = b, \\ f(X + a) - f(Y + a) = b. \end{cases}$$

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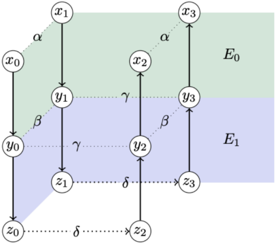
- The boomerang uniformity of the function f, denoted by $\mathcal{B}_f$, is then defined as the maximum of $\mathcal{B}_f(a, b)$, where $a, b \in \mathbb{F}_q^*$.



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- In 2021, Hasan et al. showed that for non-permutations, the differential uniformity is not necessarily smaller than the boomerang uniformity

Monomials with known boomerang uniformity in odd characteristic

	p	d	Condition	$\mathcal{B}_f$	Is PP?
C_1	3	$\frac{p^n+3}{2}$	n odd	3	Yes
C_2	$p > 2$	$p^m - 1$	$n = 2m, p \not\equiv 2 \pmod 3$	2	No
C_3	$p > 2$	$\frac{(p^m+3)(p^m-1)}{2}$	$n = 2m$	2	No
C_4	$p > 2$	$\frac{p^n-3}{2}$	$p^n \equiv 3 \pmod 4$	≤ 6	No
C_5	$p > 2$	$p^n - 2$	any n	≤ 5	Yes
C_6	$p > 2$	$k(p^m - 1)$	$n = 2m, \ \gcd(k, p^m + 1) = 1$	2	No

Table: Monomials X^d over $\mathbb{F}_{p^n}$ with known boomerang uniformity.

Our Contribution

- We consider the class of power maps $f(X) = X^{\frac{p^n+3}{2}} \in \mathbb{F}_{p^n}[X]$

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- Helleseth and Sandberg considered the differential uniformity of this class of power maps and showed that its differential uniformity Δ_f is given by

$$\Delta_f \leq \begin{cases} 1 & \text{if } p = 3 \text{ and } n \text{ is even}, \\ 3 & \text{if } p \neq 3 \text{ and } p^n \equiv 1 \pmod 4, \\ 4 & \text{otherwise.} \end{cases}$$

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- It is easy to see that when $p^n \equiv 3 \pmod{4}$ then f is a permutation.
- We considered the boomerang uniformity of the power permutation $X^{\frac{p^n+3}{2}}$, where $p^n \equiv 3 \pmod{4}$ for all $p > 3$ and showed that the boomerang uniformity is ≤ 23

Our Contribution

- Moreover, we also obtained the compositional inverse of this power permutation

Theorem

Let $q \equiv 3 \pmod 4$ then the compositional inverse of the power permutation $f(X) = X^{\frac{q+3}{2}}$ is given by

$$f^{-1}(X) = \begin{cases} X^{\frac{q+1}{4}} & \text{if } p \equiv 3 \pmod 8, \\ X^{\frac{3q-1}{4}} & \text{if } p \equiv 7 \pmod 8. \end{cases}$$

Our Contribution

- We also determined the algebraic degree of the compositional inverse

Theorem

Let $p \equiv 3 \pmod 4$ *and* $n = 2m+1$ *for some non-negative integer* m *then the algebraic degree of the inverse of the power permutation* $f(X) = X^{\frac{p^n+3}{2}}$ *is*

$$\begin{cases} \frac{(4m+1)p-4m+1}{4} & \text{if } p \equiv 3 \pmod 8, \\ \frac{(4m+3)p-4m-1}{4} & \text{if } p \equiv 7 \pmod 8. \end{cases}$$

Future Directions

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Thank you for your attention!