A family of optimal linear codes from simplicial complexes

Zhao Hu, Zhexin Wang, Nian Li, Xiangyong Zeng, and Xiaohu Tang

Abstract

In this paper, we construct a large family of projective linear codes over $\mathbb{F}_q$ from the general simplicial complexes of $\mathbb{F}_q^m$ via the defining-set construction, which generalizes the results of [IEEE Trans. Inf. Theory 66(11):6762-6773, 2020]. The parameters and weight distribution of this class of codes are completely determined. By using the Griesmer bound, we give a necessary and sufficient condition such that the codes are Griesmer codes and a sufficient condition such that the codes are distance-optimal. For a special case, we also present a necessary and sufficient condition for the codes to be near Griesmer codes. Moreover, by discussing the cases of simplicial complexes with one, two and three maximal elements respectively, many infinite families of optimal linear codes with few weights over $\mathbb{F}_q$ are obtained, including Griesmer codes, near Griesmer codes and distance-optimal codes.

Index Terms

Optimal linear code, Simplicial complex, Griesmer code, Near Griesmer code, Weight distribution

I. Introduction

Let $\mathbb{F}_{q^m}$ be the finite field with q^m elements and $\mathbb{F}_{q^m}^* = \mathbb{F}_{q^m} \setminus \{0\}$, where q is a power of a prime p and m is a positive integer. An $[n,k,d]$ linear code $\mathcal{C}$ over $\mathbb{F}_q$ is a k-dimensional subspace of $\mathbb{F}_q^n$ with minimum (Hamming) distance d. An $[n,k,d]$ linear code $\mathcal{C}$ over $\mathbb{F}_q$ is called distance-optimal if no $[n,k,d+1]$ code exists (i.e., $\mathcal{C}$ has the largest minimum distance for given n and k) and it is called almost distance-optimal if there exists an $[n,k,d+1]$ distance-optimal code. An $[n,k,d]$ linear code $\mathcal{C}$ is called optimal (resp. almost optimal) if its parameters n, k and d (resp. $d+1$) meet any bound on linear codes with equality [8]. The Griesmer bound [7], [14] for an $[n,k,d]$ linear code $\mathcal{C}$ over $\mathbb{F}_q$ is given by

$$n \geq g(k,d) := \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil,$$

where $\lceil \cdot \rceil$ denotes the ceiling function. An $[n,k,d]$ linear code $\mathcal{C}$ is called a Griesmer code (resp. near Griesmer code) if its parameters n (resp. $n-1$), k and d achieve the Griesmer bound. Griesmer codes have been an interesting topic of study for many years due to not only their optimality but also their geometric applications [4], [5]. In coding theory, it's a fundamental problem to construct (distance-)optimal codes.

Recently, constructing optimal or good linear codes from mathematical objects attracts much attention and many attempts have been made in this direction. In the various kinds of mathematical objects, simplicial complexes (which are certain subsets of $\mathbb{F}_q^m$ with good algebraic structure) are really useful to construct optimal or good linear codes. The investigation of constructing linear codes from simplicial complexes, to the best of our knowledge, first appeared in [3] (in 2018), in which Chang and Hyun constructed the first infinite family of binary minimal linear codes violating the Ashikhmin-Barg condition [2] by employing simplicial complexes of $\mathbb{F}_2^m$ with two maximal elements. In 2020, Hyun et al. [10]

Z. Hu, X. Zeng and X. Tang are with the Hubei Key Laboratory of Applied Mathematics, Faculty of Mathematics and Statistics, Hubei University, Wuhan, 430062, China. Z. Wang and N. Li are with the Hubei Key Laboratory of Applied Mathematics, School of Cyber Science and Technology, Hubei University, Wuhan, 430062, China. X. Tang is also with the Information Coding & Transmission Key Lab of Sichuan Province, CSNMT Int. Coop. Res. Centre (MoST), Southwest Jiaotong University, Chengdu, 610031, China. Email: zhao.hu@aliyun.com, zhexin.wang@aliyun.com, nian.li@hubu.edu.cn, xiangyongzeng@aliyun.com, xhutang@swjtu.edu.cn

*The full version of this paper has been submitted to the journal IEEE Transactions on Information Theory

constructed infinite families of optimal binary linear codes from the general simplicial complexes of $\mathbb{F}_2^m$ via the defining-set construction. Later, by using simplicial complexes of $\mathbb{F}_2^m$ with one maximal element, several classes of optimal or good binary linear codes with few weights were derived in [11], [16], [18] via different construction approaches. Shortly after, simplicial complexes of $\mathbb{F}_2^m$ with one and two maximal elements were utilized to construct quaternary optimal linear codes in [17], [19] by studying new defining sets of $\mathbb{F}_4^m$. Recently, some researchers also concentrated on linear codes constructed from simplicial complexes of $\mathbb{F}_q^m$ with $q > 2$. Hyun et al. [9] first defined the simplicial complexes of $\mathbb{F}_p^m$ for an odd prime p in 2019, and after that several classes of optimal p-ary few-weight linear codes were constructed in [9], [13], [15] by using different simplicial complexes of $\mathbb{F}_p^m$ with one maximal element. Later, Pan and Liu [12] defined the simplicial complexes of $\mathbb{F}_3^m$ in another way and presented three classes of few-weight ternary codes with good parameters from their defined simplicial complexes of $\mathbb{F}_3^m$ with one and two maximal elements.

In this paper, we first define the simplicial complexes of $\mathbb{F}_q^m$ for any prime power q (see the details in next section) in a different way from the definitions given by [9], [12] for a prime p and $q = 3$ resepctively. Then we employ the general simplicial complexes of $\mathbb{F}_q^m$ to construct projective linear codes $\mathcal{C}$ over $\mathbb{F}_q$ via the defining-set construction. We completely determine the parameters and weight distribution of $\mathcal{C}$. Moreover, we characterize the optimality of this family of linear codes, which shows that many (distance-)optimal codes can be produced from this construction. In addition, by studying the three cases of simplicial complexes of $\mathbb{F}_q^m$ with one, two and three maximal elements respectively, it shows that infinite families of optimal linear codes with few weights are produced from our construction, including Griesmer codes, near Griesmer codes and distance-optimal codes.

II. Preliminaries

In this section, we present some preliminaries which will be used for the subsequent sections.

Here we introduce the concept of simplicial complexes of $\mathbb{F}_q^m$, where q can be any prime power. For two vectors $u = (u_1, u_2, \ldots, u_m)$ and $v = (v_1, v_2, \ldots, v_m)$ in $\mathbb{F}_q^m$, we say that u covers v, denoted $v \preceq u$, if $\mathrm{Supp}(v) \subseteq \mathrm{Supp}(u)$, where $\mathrm{Supp}(u) = \{1 \le i \le m : u_i \ne 0\}$ is the support of u. A subset Δ of $\mathbb{F}_q^m$ is called a simplicial complex if $u \in \Delta$ and $v \preceq u$ imply $v \in \Delta$. An element u in Δ with entries 0 or 1 is said to be maximal if there is no element $v \in \Delta$ such that $\mathrm{Supp}(u)$ is a proper subset of $\mathrm{Supp}(v)$. For a simplicial complex $\Delta \subseteq \mathbb{F}_q^m$, let $\mathcal{F} = \{F_1, F_2, \ldots, F_h\}$ be the set of maximal elements of Δ, where h is the number of maximal elements in Δ and F_i's are maximal elements of Δ. Let $A_i = \mathrm{Supp}(F_i)$ for $1 \le i \le h$, which implies $A_i \subseteq [m] := \{1, 2, \ldots, m\}$. Note that $A_i \setminus A_j \ne \emptyset$ for any $1 \le i \ne j \le h$ by the definition. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_h\}$ be the set of supports of maximal elements of Δ, and $\mathcal{A}$ be called the support of Δ, denoted $\mathrm{Supp}(\Delta) = \mathcal{A}$. Then one can see that a simplicial complex Δ is uniquely generated by $\mathcal{A}$, denoted $\Delta = \langle \mathcal{A} \rangle$. Notice that both the set of maximal elements $\mathcal{F}$ and the support $\mathcal{A}$ of Δ are unique for a fixed simplicial complex Δ. For any set $\mathcal{B}$ consisting of some subsets of $[m]$, we say that a simplicial complex Δ of $\mathbb{F}_q^m$ is generated by $\mathcal{B}$, denoted $\Delta = \langle \mathcal{B} \rangle$, if Δ is the smallest simplicial complex of $\mathbb{F}_q^m$ containing every element in $\mathbb{F}_q^m$ with the support $B \in \mathcal{B}$.

Notice that the above definition of simplicial complexes of $\mathbb{F}_q^m$ is a generalization of the original definition of simplicial complexes of $\mathbb{F}_2^m$ [1], [10], and it is different from the two definitions presented in [9] for $\mathbb{F}_p^m$ and in [12] for $\mathbb{F}_3^m$.

We will construct a large family of linear code from simplicial complexes of $\mathbb{F}_q^m$ via the defining-set construction in this paper. In 2007, Ding and Niederreiter [6] introduced a nice and generic way to construct linear codes via trace functions. Let $D \subset \mathbb{F}_{q^m}$ and define

$$\mathcal{C}_D = \{c_a = (\mathrm{Tr}_q^{q^m}(ax))_{x \in D} : a \in \mathbb{F}_{q^m}\}. \tag{1}$$

Then $\mathcal{C}_D$ is a linear code over $\mathbb{F}_q$ of length $n := |D|$. The set D is called the defining set of $\mathcal{C}_D$ and the above construction is accordingly called the defining-set construction.

The following notation will be used frequently in this paper. Let $0 \leq T < q^{m-1}$ be an integer. Then T can be uniquely written as $T = \sum_{j=0}^{m-2} t_j q^j$, where $0 \leq t_j \leq q-1$ is an integer for $0 \leq j \leq m-2$. Let $v(T)$ (resp. $u(T)$) denote the smallest (resp. largest) integer in the set $\{0 \leq j \leq m-2 : t_j \neq 0\}$ and $\ell(T) = \sum_{j=0}^{m-1} t_j$.

III. The projective linear codes over $\mathbb{F}_q$ from the general simplicial complexes

Let Δ be a simplicial complex of $\mathbb{F}_{q^m}$, and Δ^c be the complement of Δ, namely, $\Delta^c = \mathbb{F}_q^m \backslash \Delta$. Notice that if $x \in \Delta$, then $yx \in \Delta$ for any $y \in \mathbb{F}_q^*$ due to the definition of simplicial complexes. Hence for any simplicial complex Δ, Δ^c can be expressed as

$$\Delta^c = \mathbb{F}_q^* \overline{\Delta}^c = \{yz : y \in \mathbb{F}_q^* \text{ and } z \in \overline{\Delta}^c\}$$

where $z_i/z_j \notin \mathbb{F}_q^*$ for distinct elements z_i and z_j in $\overline{\Delta}^c$, and clearly $|\overline{\Delta}^c| = |\Delta^c|/(q-1)$.

In this section, we investigate the projective codes $\mathcal{C}_{\overline{\Delta}^c}$ defined as in (1).

Theorem 1. *Let Δ be a simplicial complex of $\mathbb{F}_{q^m}$ with the support $\mathcal{A} = \{A_1, A_2, \ldots, A_h\}$, where $1 \leq |A_1| \leq |A_2| \leq \cdots \leq |A_h| < m$. Assume that $A_i \backslash (\cup_{1 \leq j \leq h, j \neq i} A_j) \neq \emptyset$ for any $1 \leq i \leq h$ and $q^m > \sum_{1 \leq i \leq h} q^{|A_i|}$. Denote $T = \sum_{1 \leq i \leq h} q^{|A_i|-1}$. Let $\mathcal{C}_{\overline{\Delta}^c}$ be defined as in (1). Then*

1) *$\mathcal{C}_{\overline{\Delta}^c}$ has parameters $[(q^m - |\Delta|)/(q-1), m, q^{m-1} - T]$, where $|\Delta| = \sum_{\emptyset \neq S \subseteq \mathcal{A}} (-1)^{|S|-1} q^{|\cap S|}$ and $\cap S$ is defined as $\cap S = \cap_{A \in S} A$.*
2) *$\mathcal{C}_{\overline{\Delta}^c}$ is a Griesmer code if and only if $|A_i \cap A_j| = 0$ for any $1 \leq i < j \leq h$ and at most $q-1$ of $|A_i|$'s are the same.*
3) *$\mathcal{C}_{\overline{\Delta}^c}$ is distance-optimal if $|\Delta| - 1 + (q-1)(v(T)+1) > qT - \ell(T)$.*
4) *$\mathcal{C}_{\overline{\Delta}^c}$ has the following weight enumerator*

$$\sum_{\emptyset \neq R \subseteq \Omega} |\Psi_R| z^{q^{m-1} - \sum_{S \in R} (-1)^{|S|-1} q^{|\cap S|-1}} + (q^{m - |\cup_{i=1}^h A_i|} - 1) z^{q^{m-1}} + 1$$

where $\Omega = \{S : S \subseteq \mathcal{A}, S \neq \emptyset\}$ and

$$|\Psi_R| = q^{m - |\cup_{S \in \Omega \backslash R} (\cap S)|} - \sum_{\emptyset \neq E \subseteq R} (-1)^{|E|-1} q^{m - |(\cup_{L \in E} (\cap L)) \cup (\cup_{S \in \Omega \backslash R} (\cap S))|}.$$

Remark 1. *Note that $qT - |\Delta| = \sum_{S \subseteq \mathcal{A}, |S| \geq 2} (-1)^{|S|-1} q^{|\cap S|}$ whose value heavily relies on those of $|A_i \cap A_j|$ for $1 \leq i < j \leq h$. By the definition, $v(T) \geq |A_1|$ and $\ell(T) \leq h$. Thus the condition in 3) of Theorem 1 can be easily satisfied if $|A_1|$ is large enough and $|A_i \cap A_j|$'s are small enough.*

Remark 2. *The given formula in 4) of Theorem 1 to compute the weight distribution of $\mathcal{C}_{\overline{\Delta}^c}$ is completely computable for a given Δ with support $\mathcal{A} = \{A_1, A_2, \ldots, A_h\}$ although the expression seems not so simple. Thus we say that the weight distribution of $\mathcal{C}_{\overline{\Delta}^c}$ is completely determined in Theorem 1.*

In the following corollary, we take a more in-depth discussion on the case that $|A_i \cap A_j| = 0$ for all $1 \leq i < j \leq h$ for the code $\mathcal{C}_{\overline{\Delta}^c}$ in Theorem 1.

Corollary 1. *Let Δ be a simplicial complex of $\mathbb{F}_{q^m}$ with the support $\mathcal{A} = \{A_1, A_2, \ldots, A_h\}$, where $1 \leq |A_1| \leq |A_2| \leq \cdots \leq |A_h| < m$. Assume that $|A_i \cap A_j| = 0$ for $1 \leq i < j \leq h$. Denote $T = \sum_{1 \leq i \leq h} q^{|A_i|-1}$. Let $\mathcal{C}_{\overline{\Delta}^c}$ be defined as in (1). Then $\mathcal{C}_{\overline{\Delta}^c}$ is an at most 2^h-weight $[(q^m - \sum_{i=1}^h q^{|A_i|} + h - 1)/(q-1), m, q^{m-1} - T]$ linear code with weight enumerator*

$$\sum_{\emptyset \neq R \subseteq [h]} (q^{m - \sum_{i \in [h] \backslash R} |A_i|} - \sum_{\emptyset \neq E \subseteq R} (-1)^{|E|-1} q^{m - \sum_{i \in E} |A_i| - \sum_{i \in [h] \backslash R} |A_i|}) z^{q^{m-1} - \sum_{i \in R} q^{|A_i|-1}} + (q^{m - \sum_{i=1}^h |A_i|} - 1) z^{q^{m-1}} + 1.$$

Moreover, we have the followings:

1) *$\mathcal{C}_{\overline{\Delta}^c}$ is a Griesmer code if and only if at most $q-1$ of $|A_i|$'s are the same;*
2) *$\mathcal{C}_{\overline{\Delta}^c}$ is a near Griesmer code if and only if $\ell(T) = h - (q-1)$; and*

3) $\mathcal{C}_{\overline{\Delta}^c}$ *is distance-optimal if* $\ell(T)+(q-1)(v(T)+1)>h$. *Specially, when* $|A_i|=\varepsilon$ *for* $1\leq i\leq h$, *where* ε *is a positive integer, it is distance-optimal if* $\ell(h)+(q-1)(v(h)+\varepsilon)>h$.

Remark 3. *The Griesmer codes in Corollary 1 (or Theorem 1) are indeed the Solomon-Stiffler codes. Definitely, for the other cases (not the Griesmer codes), our codes* $\mathcal{C}_{\overline{\Delta}^c}$ *in Corollary 1 and Theorem 1 are different from the Solomon-Stiffler codes.*

Remark 4. *Notice that the condition in 2) of Corollary 1 can be easily satisfied by selecting proper* A_i*'s. Moreover, the condition* $\ell(T)+(q-1)(v(T)+1)>h$ *for* $\mathcal{C}_{\overline{\Delta}^c}$ *to be distance-optimal can be easily satisfied if* $|A_1|$ *is large enough since* $1\leq\ell(T)\leq h$ *and* $v(T)\geq|A_1|$, *and consequently many distance-optimal linear codes can be produced in Corollary 1 besides (near) Griesmer codes.*

Next, we give more explicit results on the cases $h=1,2,3$ of Theorem 1.

Corollary 2. *Let* Δ *be a simplicial complex of* $\mathbb{F}_{q^m}$ *with exactly one maximal element and its support is* $\{A\}$ *with* $A\subseteq[m]$ *and* $1\leq|A|<m$. *Then* $\mathcal{C}_{\overline{\Delta}^c}$ *defined by* (1) *is a* 2*-weight* $[(q^m-q^{|A|})/(q-1),m,q^{m-1}-q^{|A|-1}]$ *linear code with weight distribution*

Weight w	*Multiplicity* A_w
0	1
q^{m-1}	$q^{m-\|A\|}-1$
$q^{m-1}-q^{\|A\|-1}$	$q^m-q^{m-\|A\|}$

and it is a Griesmer code.

Corollary 3. *Let* Δ *be a simplicial complex of* $\mathbb{F}_{q^m}$ *with the support* $\mathcal{A}=\{A_1,A_2\}$, *where* $1\leq|A_1|\leq|A_2|<m$. *Assume that* $q^m>q^{|A_1|}+q^{|A_2|}$. *Let* $T=q^{|A_1|-1}+q^{|A_2|-1}$. *Then* $\mathcal{C}_{\overline{\Delta}^c}$ *defined by* (1) *is an at most* 5*-weight* $[(q^m-q^{|A_1|}-q^{|A_2|}+q^{|A_1\cap A_2|})/(q-1),m,q^{m-1}-q^{|A_1|-1}-q^{|A_2|-1}]$ *linear code and its weight distribution is given by*

Weight w	*Multiplicity* A_w
0	1
q^{m-1}	$q^{m-\|A_1\cup A_2\|}-1$
$q^{m-1}-q^{\|A_2\|-1}$	$q^{m-\|A_1\|}-q^{m-\|A_1\cup A_2\|}$
$q^{m-1}-q^{\|A_1\|-1}$	$q^{m-\|A_2\|}-q^{m-\|A_1\cup A_2\|}$
$q^{m-1}-q^{\|A_1\|-1}-q^{\|A_2\|-1}$	$q^{m-\|A_1\cap A_2\|}-q^{m-\|A_1\|}-q^{m-\|A_2\|}+q^{m-\|A_1\cup A_2\|}$
$q^{m-1}-q^{\|A_1\|-1}-q^{\|A_2\|-1}+q^{\|A_1\cap A_2\|-1}$	$q^m-q^{m-\|A_1\cap A_2\|}$

Moreover, we have the followings:

1) *When* $|A_1\cap A_2|=0$ *and* $|A_1|=|A_2|$, $\mathcal{C}_{\overline{\Delta}^c}$ *is a near Griesmer code (also distance-optimal) if* $q=2$ *and it is a Griesmer code if* $q>2$. *It reduces to a* 3*-weight code in this case.*
2) *When* $|A_1\cap A_2|=0$ *and* $|A_1|<|A_2|$, $\mathcal{C}_{\overline{\Delta}^c}$ *is a Griesmer code and it reduces to a* 4*-weight code.*
3) *When* $|A_1\cap A_2|>0$ *and* $|A_1|=|A_2|$, $\mathcal{C}_{\overline{\Delta}^c}$ *is distance-optimal if* $\ell(T)+(q-1)(v(T)+1)>q^{|A_1\cap A_2|}+1$ *and it reduces to a* 4*-weight code. Specially,* $\mathcal{C}_{\overline{\Delta}^c}$ *is a near Griesmer code if* $q>2$ *and* $|A_1\cap A_2|=1$.
4) *When* $|A_1\cap A_2|>0$ *and* $|A_1|<|A_2|$, $\mathcal{C}_{\overline{\Delta}^c}$ *is distance-optimal if* $(q-1)|A_1|+1>q^{|A_1\cap A_2|}$. *Specially,* $\mathcal{C}_{\overline{\Delta}^c}$ *is a near Griesmer code if* $|A_1\cap A_2|=1$.

Corollary 4. *Let* Δ *be a simplicial complex of* $\mathbb{F}_{q^m}$ *with the support* $\mathcal{A}=\{A_1,A_2,A_3\}$, *where* $1\leq|A_1|\leq|A_2|\leq|A_3|<m$. *Assume that* $A_i\backslash(\cup_{1\leq j\leq 3,j\neq i}A_j)\neq\emptyset$ *for any* $1\leq i\leq 3$, *and* $q^m>\sum_{1\leq i\leq 3}q^{|A_i|}$. *Let* $T=\sum_{1\leq i\leq 3}q^{|A_i|-1}$. *Then* $\mathcal{C}_{\overline{\Delta}^c}$ *defined by* (1) *is a* $[(q^m-|\Delta|)/(q-1),m,q^{m-1}-T]$ *linear code, where* $|\Delta|=\sum_{i=1}^{3}q^{|A_i|}-\sum_{1\leq i<j\leq 3}q^{|A_i\cap A_j|}+q^{|A_1\cap A_2\cap A_3|}$. *Moreover, we have the followings:*

1) $\mathcal{C}_{\overline{\Delta}^c}$ *is a Griesmer code if and only if* $|A_i\cap A_j|=0$ *for* $1\leq i<j\leq 3$ *and at most* $q-1$ *of* $|A_i|$*'s are the same (which always holds for* $q>3$*).*

2) $\mathcal{C}_{\Delta^c}$ *is a near Griesmer code if one of the followings holds: i)* $|A_i \cap A_j| = 1$ *for only one element* (i, j) *in the set* $\{(i, j) : 1 \le i < j \le 3\}$ *and* $|A_i \cap A_j| = 0$ *for the other two* (i, j)*'s, and at most* $q-1$ *of* $|A_i|$*'s are the same; ii)* $q = 3$*,* $|A_i \cap A_j| = 0$ *for* $1 \le i < j \le 3$*, and* $|A_1| = |A_2| = |A_3|$*; and iii)* $q = 2$*,* $|A_i \cap A_j| = 0$ *for* $1 \le i < j \le 3$*, and* $|A_1| = |A_2| < |A_3| - 1$ *or* $|A_1| \le |A_2| = |A_3|$*.*
3) $\mathcal{C}_{\Delta^c}$ *is distance-optimal if* $(q-1)(v(T)+1) + \ell(T) - 1 > \sum_{1 \le i < j \le 3} q^{|A_i \cap A_j|} - q^{|A_1 \cap A_2 \cap A_3|}$*.*

Remark 5. *The weight distribution of* $\mathcal{C}_{\Delta^c}$ *in Corollary 4 also can be determined by the formula in 4) of Theorem 1, which is at most* 19*-weight.*

IV. Concluding remarks

The main contributions of this paper are summarized as follows:

- We constructed a large family of projective linear codes $\mathcal{C}_{\Delta^c}$ over $\mathbb{F}_q$ from the general simplicial complexes Δ of $\mathbb{F}_q^m$ via the defining-set construction. This totally extends the results of [10] from $\mathbb{F}_2$ to $\mathbb{F}_q$. To the best of our knowledge, this paper is the first to study linear codes over $\mathbb{F}_q$ constructed from the general simplicial complexes of $\mathbb{F}_q^m$ for a prime power $q > 2$.
- The parameters and weight distribution of $\mathcal{C}_{\Delta^c}$ were completely determined (see Theorem 1) in this paper. Thus this paper also determines the weight distribution of the binary codes constructed from the general simplicial complexes of $\mathbb{F}_2^m$ in [10, Theorem IV.6], in which the weight distribution of the binary codes were studied only for the case of simplicial complexes of $\mathbb{F}_2^m$ with two maximal elements. Moreover, as a byproduct, the weight distributions of the Solomon-Stiffler codes are determined in Corollary 1 for the case that the corresponding subspaces in $\mathbb{F}_q^m$ of the projective subspaces U_i are spanned by some subsets of a certain basis of $\mathbb{F}_q^m$.
- By using the Griesmer bound, we gave a necessary and sufficient condition such that $\mathcal{C}_{\Delta^c}$ is a Griesmer code and a sufficient condition such that $\mathcal{C}_{\Delta^c}$ is distance-optimal. In addition, we also presented a necessary and sufficient condition for $\mathcal{C}_{\Delta^c}$ to be a near Griesmer code in a special case. This shows that many infinite families of (distance-)optimal linear codes can be produced from our construction.
- By studying the cases of the simplicial complexes Δ with one, two and three maximal elements respectively, we derived infinite families of optimal linear codes with few weights over $\mathbb{F}_q$ including Griesmer codes, near Griesmer codes and distance-optimal codes.

References

[1] M. Adamaszek, Face numbers of down-sets, Amer. Math. Monthly 122(4) (2015), pp. 367-370.
[2] A. Ashikhmin, A. Barg, Minimal vectors in linear codes, IEEE Trans. Inf. Theory 44(5) (1998), pp. 2010-2017.
[3] S. Chang, J.Y. Hyun, Linear codes from simplicial complexes, Des. Codes Cryptogr. 86 (2018), pp. 2167-2181.
[4] C. Ding, Codes from Difference Sets, World Scientific, Singapore (2015).
[5] C. Ding, Designs from Linear Codes, World Scientific, Singapore (2018).
[6] C. Ding, H. Niederreiter, Cyclotomic linear codes of order 3, IEEE Trans. Inf. Theory 53(6) (2007), pp. 2274-2277.
[7] J.H. Griesmer, A bound for error correcting codes, IBM J. Res. Dev. 4 (1960), pp. 532-542.
[8] W. Huffman, V. Pless, Fundamentals of error-correcting codes, Cambridge University Press (1997).
[9] J.Y. Hyun, H.K. Kim, M. Na, Optimal non-projective linear codes constructed from down-sets, Discrete Appl. Math. 254 (2019), pp. 135-145.
[10] J.Y. Hyun, J. Lee, Y. Lee, Infinite families of optimal linear codes constructed from simplicial complexes, IEEE Trans. Inf. Theory 66(11) (2020), pp. 6762-6773.
[11] X. Li, M. Shi, A new family of optimal binary few-weight codes from simplicial complexes, IEEE Communications Letters 25(4) (2021), pp. 1048-1051.
[12] Y. Pan, Y. Liu, New classes of few-weight ternary codes from simplicial complexes, AIMS Math. 7(3) (2022), pp. 4315-4325.
[13] M. Shi, X. Li, Two classes of optimal p-ary few-weight codes from down-sets, Discret. Appl. Math. 290 (2021), pp. 60-67.
[14] G. Solomon, J.J. Stiffler, Algebraically punctured cyclic codes, Inform. and Control 8 (1965), pp. 170-179.
[15] Y. Wu, J.Y. Hyun, Few-weight codes over $\mathbb{F}_p + u\mathbb{F}_p$ associated with down sets and their distance optimal Gray image, Discret. Appl. Math. 283 (2020), pp. 315-322.
[16] Y. Wu, Y. Lee, Binary LCD codes and self-orthogonal codes via simplicial complexes, IEEE Communications Letters 24(6) (2020), pp. 1159-1162.
[17] Y. Wu, C. Li, F. Xiao, Quaternary linear codes and related binary subfield codes, IEEE Trans. Inf. Theory 68(5) (2022), pp. 3070-3080.
[18] Y. Wu, X. Zhu, Q. Yue, Optimal few-weight codes from simplicial complexes, IEEE Trans. Inf. Theory 66(6) (2020), pp. 3657-3663.
[19] X. Zhu, Y. Wei, Few-weight quaternary codes via simplicial complexes, AIMS Math. 6(5) (2021), pp. 5124-5132.