Equations over the finite field $\mathbb{F}_{2^n}$

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Outline

1. On some equations in $\mathbb{F}_{2^n}$
2. Solving $x^{2^k+1} + x + a = 0$ in $\mathbb{F}_{2^n}$ with $(n, k) = 1$ [joint work with Kwang Ho Kim]
   1. Motivation
3. Preliminaries
4. The two related problems for solving $x^{2^k+1} + x + a = 0$ in $\mathbb{F}_{2^n}$ with $(n, k) = 1$
5. Solving the two problems
6. The solution of the equation $x^{2^k+1} + x + a = 0$ in $\mathbb{F}_{2^n}$
7. Conclusions
\[ F_{2^n} \text{ the finite field of order } 2^n. \]

The *absolute trace* over \( F_2 \) of an element \( x \in F_{2^n} \) equals
\[ \text{Tr}^n_1(x) = \sum_{i=0}^{n-1} x^{2^i}. \]
A fundamental equation

Let $q$ be a power of 2.

- The equation $x^q - x = 0$ admits $\mathbb{F}_q$ as set of solutions.
- Finding the solutions in $\mathbb{F}_q$ of an equation $P(x) = 0$ over $\mathbb{F}_q$ is equivalent to finding the solutions of the equation $(P(x), x^q - x) = 0$. The number of solutions equals the degree of $(P(x), x^q - x)$. 

Equations in $\mathbb{F}_q$.

**Equation of degree 1**
The equation $ax + b = 0$, $a \neq 0$, admits one solution $-b/a$, in $\mathbb{F}_q$ in any field.
Equations in $\mathbb{F}_{2^n}$

**Equations of degree 2**

- A necessary condition for the existence of a solution $x$ in $\mathbb{F}_{2^n}$ of the equation $x^2 + x = \beta$ is that $\text{Tr}^n_1(\beta) = 0$.

**Theorem**

The solutions of the equation $x^2 + x = \beta$ are $x = \sum_{j=1}^{n-1} \beta^{2j} (\sum_{k=0}^{j-1} c^{2k})$ and $x = 1 + \sum_{j=1}^{n-1} \beta^{2j} (\sum_{k=0}^{j-1} c^{2k})$, where $c$ is any (fixed) element such that $\text{Tr}^n_1(c) = 1$.

- $ax^2 + bx + c = 0$, $a \neq 0$ is equivalent to $\left( \frac{ax}{b} \right)^2 + \frac{ax}{b} = \frac{ac}{b^2}$.

- The equation $ax^2 + bx + c = 0$ of degree 2 reduces to solving the equation $x^2 + x = \beta$.
On the equation $x + x^{2^k} = b$

**Equation $x + x^{2^k} = b$ in $\mathbb{F}_{2^n}$**

Define $S_{n,k}(x) = \sum_{i=0}^{n-1} x^{2^{ki}}$ and $M = \{ \zeta \in \mathbb{F}_{2^n} \mid \zeta^{2^n+1} = 1 \}$. Then,

**Proposition (K. H. Kim - SM 2019)**

Let $(k, n) = 1$ and $k$ odd. Let $\zeta$ be an element of $M \setminus \{1\}$. Then, for any $b \in \mathbb{F}_{2^n}^*$, we have

$$\{ x \in \mathbb{F}_{2^n} \mid x + x^{2^k} = b \} = S_{n,k} \left( \frac{b}{\zeta + 1} \right) + \mathbb{F}_2$$
On the equation \( x + x^{2^k} = b \)

**Proof:**
Set \( q = 2^k \). As it was assumed that \( k \) is odd and \((n, k) = 1\), it holds \((2n, k) = 1\) and so the linear mapping \( x \in \mathbb{F}_{2^{2n}} \longmapsto x + x^q \) has kernel of dimension 1, i.e. the equation \( x + x^q = b \) has at most 2 solutions in \( \mathbb{F}_{2^{2n}} \). Since \( S_{n,k}(x) + (S_{n,k}(x))^q = x + x^q \), we have

\[
S_{n,k} \left( \frac{b}{\zeta + 1} \right) + \left( S_{n,k} \left( \frac{b}{\zeta + 1} \right) \right)^q + b = \frac{b}{\zeta + 1} + \left( \frac{b}{\zeta + 1} \right)^{q^n} + b
\]

\[
= \frac{b}{\zeta + 1} + \frac{b}{\zeta^{q^n} + 1} + b
\]

\[
= \frac{b}{\zeta + 1} + \frac{b}{1/\zeta + 1} + b
\]

\[
= 0
\]

and thus really \( S_{n,k} \left( \frac{b}{\zeta + 1} \right), S_{n,k} \left( \frac{b}{\zeta + 1} \right) + 1 \in \mathbb{F}_{2^{2n}} \) are the \( \mathbb{F}_{2^{2n}} \)-solutions of the equation \( x + x^q = b \).
Equations in $\mathbb{F}_{2^n}$

Equation of degree 3: $x^3 + ax + b = 0$

**Theorem (Berlekamp-Rumsey-Solomon 1967-Williams 1975)**

Let $t_1$ and $t_2$ denote the roots of $t^2 + bt + a^3$ in $\mathbb{F}_{2^n}$, where $a \in \mathbb{F}_{2^n}$, $b \in \mathbb{F}_{2^n}^*$. Let $f(x) = x^3 + ax + b$ over $\mathbb{F}_{2^n}$. Then

- $f$ has three zeros in $\mathbb{F}_{2^n}$ if and only if $\text{Tr}_1^n \left( \frac{a^3}{b^2} + 1 \right) = 0$ and $t_1, t_2$ are cubes in $\mathbb{F}_{2^n}$ ($n$ even), $\mathbb{F}_{2^{2n}}$ ($n$ odd).

- $f$ has exactly one zero in $\mathbb{F}_{2^n}$ if and only if $\text{Tr}_1^n \left( \frac{a^3}{b^2} + 1 \right) = 1$.

- $f$ has no zero in $\mathbb{F}_{2^n}$ if and only if $\text{Tr}_1^n \left( \frac{a^3}{b^2} + 1 \right) = 0$ and $t_1, t_2$ are not cubes in $\mathbb{F}_{2^n}$ ($n$ even), $\mathbb{F}_{2^{2n}}$ ($n$ odd).
Let $i$ be a positive integer. Let $U$ be a multiplicative subgroup of $\mathbb{F}_{2^n}^\ast$ of order $\frac{2^n - 1}{\gcd(i, 2^n - 1)}$. The equation $x^i = a$ has:

- one solution if $a = 0$;
- no solution if $a \in \mathbb{F}_{2^n}^\ast \setminus U$;
- $\gcd(i, 2^n - 1)$ solutions if $a \in U$. 
Solving $x^{2^k+1} + x + a = 0$ has interests in

- the general theory of finite fields
- the construction of difference sets with Singer parameters [Dillon 2002];
- finding cross-correlation between $m$-sequences [Helleseth-Kholosha-Ness 2007];
- constructing error correcting codes [Bracken-Helleseth 2009];
- the context of APN functions [Budaghyan-Carlet 2006], [Bracken-Tan-Tan 2014], [Canteaut-Perrin-Tian 2019];
- constructions designs [Tang 2019];
- etc.
The $k$-subset $D$ of the group $G$ of order $v$ is a difference set with parameters $(v, k, \lambda)$ if for all nonidentity elements $g$ of $G$ the equation $g = xy^{-1}$ has exactly $\lambda$ solutions with $x$ and $y$ in $D$.

If $G$ is the multiplicative group of $\mathbb{F}_{2^m}$ of order $2^m - 1$, then the subset $D$ of $G$ is a difference set with the so-called Singer parameters if

$(v, k, \lambda) = (2^m - 1, 2^{m-1}, 2^{m-2})$ (or the complementary parameters $(2^m - 1, 2^{m-1} - 1, 2^{m-2} - 1)$).

The polynomial $x^{2^k+1} + x + a$ allows to construct difference sets with Singer parameters $(v, k, \lambda) = (2^m - 1, 2^{m-1}, 2^{m-2})$ with $m \geq 3$. 

[Dillon 2002], [Dillon-Dobbertin 2004]
\[ x^{2k+1} + x + a = 0 : \text{motivation 2} \]

**Theorem (Budaghyan-Carlet 2006)**

*Under some conditions, if* \( G(x) := x^{2i+1} + cx^{2i} + c^2 x + 1 \) *has no solution* \( x \) *such that* \( x^{2k+1} = 1 \) *the* \( F(x) = x(x^{2i} + x^{2k} + cx^{2k+i}) + x^2 (c^2 x^{2k} + bx^{2k+i}) + x^{2k+i+2k} \) *is APN on* \( \mathbb{F}_{2^k} \).

[Bracken-Tan-Tan 2014] constructed explicitly the polynomial \( G \) (when \( k \) even and 3 does not divide \( k \)).

- The polynomial \( G \) relates to the polynomial \( x^{2k+1} + x + a = 0 : \) substituting \( sx + c \) to \( x \) with \( s^{2i} = c^{2i} + c^{2k} \) we get \( G(sx + c) = s^{2i+1}(x^{2k+1} + x + a) \).
$x^{2^k+1} + x + a = 0$ : motivation 3

**Definition**

Let $s(t)$ and $v(t)$ be two binary $m$-sequences. $s(t) = \text{Tr}^m_1(\alpha^t)$ where $\alpha$ is an element of order $n = 2^m - 1$. Assume $v(t) = u(dt)$ where $u(t) = \text{Tr}^k_1(\beta^t)$ where $\beta$ is an element of order $2^{m/2} - 1$. Let $d$ such that $\gcd(d, 2^{m/2} - 1) = 1$. The cross-correlation function $C_d(\tau)$ between the two $m$-sequences $s(t)$ and $v(t)$ is defined (for $\tau = 0, 1, \cdots, 2^k - 2$) by $C_d(\tau) = \sum_{t=0}^{n-1} (-1)^{s(t)+v(t+\tau)}$.

[Helleseth-Kholosha-Ness 2007] gave a three-valued cross-correlation function between the pairs of sequences of different lengths.

**Theorem (Helleseth-Kholosha-Ness 2007)**

Let $m = 2k$ and $d(2^l + 1) \equiv 2^i \pmod{2^k - 1}$ for some odd $k$ and integer $l$ with $0 < l < k$ and $\gcd(l, k) = 1$. Then the cross-correlation function $C_d(\tau)$ has the following distribution:

$-1 - 2^{k+1}$ occurs $\frac{2^{k-1}-1}{3}$ times; $-1$ occurs $2^{k-1} - 1$ times; $-1 + 2^k$ occurs $\frac{2^{k+1}}{3}$ times.
To prove their main result above, they need to compute three exponential sums $S_i(a) = \sum_{y \in \mathbb{F}_{2^m}} (-1)^{Tr_1^m(ry^{2^l}+1)+Tr_1^k(y^{2^k}+1)}$ for $i = 0, 1$

$S_2(a) = \sum_{y \in \mathbb{F}_{2^m}} (-1)^{Tr_1^m(r^{-1}ay^{2^l}+1)+Tr_1^k(y^{2^k}+1)}$

In order to determine $S_0(a)$, they need to consider zeros in $\mathbb{F}_{2^k}$ of the affine polynomial $A_a(v) = a^2v^{2^l} + v^{2^l} + av + 1$ where $l < k$ and $(l, k) = 1$

The distribution of the zeros in $\mathbb{F}_{2^n}$ of $A_a(v) = a^2v^{2^l} + v^{2^l} + av + 1$ will determine to a large extent the distribution of their cross-correlation function.

**Theorem** (Helleseth-Kholosha-Ness 2007)

Let $M_i = \{ a \mid A_a(v) \text{ has exactly } i \text{ zeros in } \mathbb{F}_{2^n} \}$ Then $A_a(v)$ has either one, two, or four zeros in $\mathbb{F}_{2^n}$. For $i \in \{1, 2, 4\}$ we have $a \in M_i$ if and only if $x^{2^k+1} + x + a = 0$ has exactly $i - 1$ zeros in $\mathbb{F}_{2^n}$. 
The binary primitive triple-error-correcting BCH code is a cyclic code of minimum distance $d = 7$ with generator polynomial $g(x)$ having zeros $\alpha, \alpha^3$ and $\alpha^5$ where $\alpha$ is a primitive $(2^n - 1)$-root of unit in $\mathbb{F}_{2^n}$. The zero set of the code is said to be the triple $1, 3, 5$. Let $d_1 = 1$, $d_2 = 3$ and $d_3 = 5$. Then the parity-check matrix

$$H = \begin{pmatrix}
1 & \alpha^{d_1} & \alpha^{2d_1} & \ldots & \alpha^{(2^n-2)d_1} \\
1 & \alpha^{d_2} & \alpha^{2d_2} & \ldots & \alpha^{(2^n-2)d_2} \\
1 & \alpha^{d_3} & \alpha^{2d_3} & \ldots & \alpha^{(2^n-2)d_3}
\end{pmatrix}. \quad (1)$$

**THEOREM (Bracken-Helleseth 2009)**

Let \( n \) be odd and \( \gcd(k, n) = 1 \). Then the error-correcting code constructed using the zero set \( 1, 2^k + 1, 2^{3k} + 1 \) is triple-error-correcting.

Their proof shows an interesting connection to the equation of the form \( x^{2^k+1} + bx^{2^k} + cx = d \) defined on \( \mathbb{F}_{2^n} \) which has no more than three solutions when \( \gcd(k, n) = 1 \) for all \( b, c, \) and \( d \) in \( \mathbb{F}_{2^n} \) (as a consequence of a result in [Bluher 2004] on \( x^{2^k+1} + x + a = 0 \)).
\[ x^{2^k+1} + x + a = 0 : \text{motivation 5} \]

**Definition**

Let \( \mathcal{P} \) be a set of \( v \) elements and let \( \mathcal{B} \) be a set of \( k \)-subsets of \( \mathcal{P} \). Let \( t \) be a positive integer with \( t \leq k \). The pair \((\mathcal{P}, \mathcal{B})\) is called incident structure. It said to be a \( t-(v, k, \lambda) \) design if every \( t \)-subset of \( \mathcal{P} \) is contained in exactly \( \lambda \) elements of \( \mathcal{B} \).
[Tang 2019] constructed 3-designs: let $q = 2^n$ and let
\[ B_s := \{(x + 1)^s + x^s \mid x \in \mathbb{F}_q\}. \]

**Proposition (Tang 2019)**

Let $n = 3k \pm 1$ and $s = 2^{2k} - 2^k + 1$ where $i$ an even integer. Let $d = 1/s (\text{mod } 2^n - 1)$. Then the incidence structure $\left( \mathbb{F}_q, \{\pi(B_s) \mid \pi(x) = ax + b\} \right)$ is 3-design if and only if $\#\{x \in \mathbb{F}_{2^n} \mid u^d x + (1 + u^d)^{2^k + 1} + x^{2^k + 1} + 1 = 0\}$ is independent of $u \in \mathbb{F}_q \setminus \mathbb{F}_2$.

The equation $u^d x + (1 + u^d)^{2^k + 1} + x^{2^k + 1} + 1 = 0$ can be reduced to
\[ x^{2^k + 1} + x + a = 0. \]
Müller-Cohen-Matthews polynomials are defined over $\mathbb{F}_{2^n}$ as follows:

$$f_{k,d}(X) := \frac{T_k(X^c)^d}{X^{2^k}}$$

where

$$T_k(X) := \sum_{i=0}^{k-1} X^{2^i} \quad \text{and} \quad cd = 2^k + 1.$$ 

A basic property for such polynomials is:

**Theorem (1)**


Let $k$ and $n$ be two positive integers with $(k, n) = 1$.

1. If $k$ is odd, then $f_{k,2^k+1}$ is a permutation on $\mathbb{F}_{2^n}$.
2. If $k$ is even, then $f_{k,2^k+1}$ is a 2-to-1 on $\mathbb{F}_{2^n}$. 

The Dickson polynomial of the first kind of degree \( k \) in indeterminate \( x \) and with parameter \( a \in \mathbb{F}_{2^n}^* \) is

\[
D_k(x, a) = \sum_{i=0}^{[k/2]} \frac{k}{k-i} \binom{k-i}{i} a^i x^{k-2i},
\]

where \( [k/2] \) denotes the largest integer less than or equal to \( k/2 \).

In this talk, we consider only Dickson polynomials \( D_k(x, 1) \), that we shall denote \( D_k(x) \).

**Proposition**

For any positive integer \( k \) and any \( x \in \mathbb{F}_{2^n} \), we have

\[
D_k \left( x + \frac{1}{x} \right) = x^k + \frac{1}{x^k}. \tag{2}
\]
Known results about $P_a(x) := x^{2^k+1} + x + a = 0$ when $(n, k) = 1$

Let $N_a$ be the number of solutions of the equation $P_a(x) := x^{2^k+1} + x + a = 0$ in $\mathbb{F}_{2^n}$.

- In 2004: [Bluher 2004] the number of solutions $N_a$ are only 0, 1 and 3 when $(k, n) = 1$.

- In 2008: [Helleseth-Kholosha 2008] got criteria for $N_a = 1$ and an explicit expression of the unique solution when $(k, n) = 1$.

- In 2014: [Bracken-Tan-Tan 2014] presented a criterion for $N_a = 0$ when $n$ is even and $(k, n) = 1$. 
On the equation $x^{q+1} + x + a = 0; q = 2^k$

**Notation**: $q = 2^k$.
We will exploit a recent polynomial identity involving Dickson polynomials:

**Theorem (2)**

[Bluher 2016]

In the polynomial ring $\mathbb{F}_q[X, Y]$, we have the identity

$$X^{q^2-1} + \left( \sum_{i=1}^{k} Y^{q-2^i} \right) X^{q-1} + Y^{q-1} = \prod_{w \in \mathbb{F}_q^*} (D_{q+1}(wX) - Y).$$
Solving \( P_a(x) := x^{q+1} + x + a = 0 \); \( q = 2^k \)

If \( k \) is odd, since \((q - 1, 2^n - 1) = 1\), the zeros of \( P_a(x) \) are the images of the zeros of \( P_a(x^{q-1}) \) by the map \( x \mapsto x^{q-1} \).

Now \( f_{k,q+1} \) is a permutation polynomial of \( \mathbb{F}_{2^n} \) by Theorem 1. Therefore, for any \( a \in \mathbb{F}_{2^n}^* \), there exists a unique \( Y \) in \( \mathbb{F}_{2^n}^* \) such that \( a = \frac{1}{f_{k,q+1} \left( \frac{1}{Y} \right)^{\frac{q}{2}}} \). Hence, we have

\[
P_a \left( x^{q-1} \right) = x^{q^2-1} + x^{q-1} + \frac{1}{f_{k,q+1} \left( \frac{1}{Y} \right)^{\frac{q}{2}}} \tag{3}
\]

Substituting \( tx \) to \( X \) in the above identity with \( t^{q^2-q} = Y^q T_k \left( \frac{1}{Y} \right)^2 \), we get :

\[
P_a \left( x^{q-1} \right) = x^{q^2-1} + x^{q-1} + \frac{1}{f_{k,q+1} \left( \frac{1}{Y} \right)^{\frac{q}{2}}} = \frac{1}{Y^{q-1} \left( f_{k,q+1} \left( \frac{1}{Y} \right) \right)^{\frac{q}{2}}} \left( X^{q^2-1} + \left( \sum_{i=1}^{k} Y^{q-2i} \right) X^{q-1} + Y^{q-1} \right)
\]
On the equation \( x^{q+1} + x + a = 0 \); \( q = 2^k \)

By Theorem 2:

\[
X^{q^2-1} + \left( \sum_{i=1}^{k} Y^{q-2^i} \right) X^{q-1} + Y^{q-1} = \prod_{w \in \mathbb{F}_q^*} (D_{q+1} (wX) - Y)
\]

Therefore

\[
P_a (x^{q-1}) = \frac{1}{Y^{q-1} (f_{k,q+1} (\frac{1}{Y}))^{\frac{2^q}{q}}} \prod_{w \in \mathbb{F}_q^*} (D_{q+1} (wtx) - Y)
\]

When \( k \) is odd, finding the zeros of \( P_a (x^{q-1}) \) amounts to determine preimages of \( Y \) under the Dickson polynomial \( D_{q+1} \).
Solving $P_a(x) := x^{q+1} + x + a = 0 ; q = 2^k$

When $k$ is even, $f_{k,q+1}$ is 2-to-1, Fortunately, we can go back to the odd case by rewriting the equation. Indeed, for $x \in \mathbb{F}_{2^n}$,

$$P_a(x) = x^{2^k+1} + x + a = \left(x^{2^{n-k}+1} + x^{2^{n-k}} + a^{2^{n-k}}\right)^{2^k}$$

$$= \left((x + 1)^{2^{n-k}+1} + (x + 1) + a^{2^{n-k}}\right)^{2^k}$$

and so

$$\{x \in \mathbb{F}_{2^n} | P_a(x) = 0\} = \left\{x + 1 \mid x^{2^{n-k}+1} + x + a^{2^{n-k}} = 0, x \in \mathbb{F}_{2^n}\right\}.$$  \hspace{1cm} (4)

If $k$ is even, then $n - k$ is odd and we can reduce to the odd case.
Solving $P_a(x) := x^{2k+1} + x + a = 0$

We now summarize all the above discussions in the following theorem.

**Theorem (K. H. Kim - SM 2019)**

Let $k$ and $n$ be two positive integers such that $(k, n) = 1$.

1. Let $k$ be odd and $q = 2^k$. Let $Y \in \mathbb{F}_{2n}^*$ be (uniquely) defined by $a = \frac{1}{f_{k,q+1} \left( \frac{1}{Y} \right)^{\frac{2}{q}}}$. Then,

$$\{ x \in \mathbb{F}_{2n} \mid P_a(x) = 0 \} = \left\{ \frac{z^{q-1}}{Y T_k \left( \frac{1}{Y} \right)^{\frac{2}{q}}} \mid D_{q+1}(z) = Y, z \in \mathbb{F}_{2n} \right\}.$$

2. Let $k$ be even and $q' = 2^{n-k}$. Let $Y' \in \mathbb{F}_{2n}^*$ be (uniquely) defined by $a^{q'} = \frac{1}{f_{n-k,q'+1} \left( \frac{1}{Y'} \right)^{\frac{2}{q'}}}$. Then,

$$\{ x \in \mathbb{F}_{2n} \mid P_a(x) = 0 \} = \left\{ 1 + \frac{z^{q'-1}}{Y' T_{n-k} \left( \frac{1}{Y'} \right)^{\frac{2}{q'}}} \mid D_{q'+1}(z) = Y', z \in \mathbb{F}_{2n} \right\}. $$
Solving \( P_a(x) := x^{q+1} + x + a = 0 ; q = 2^k \)

we can split the problem of finding the zeros in \( \mathbb{F}_{2^n} \) of \( P_a \) into two independent problems with odd \( k \).

**Problem (1)**

For \( a \in \mathbb{F}_{2^n}^* \), find the unique element \( Y \) in \( \mathbb{F}_{2^n}^* \) such that

\[
 a^{q^2} = \frac{1}{f_{k,q+1}(\frac{1}{Y})}. \tag{5}
\]

**Problem (2)**

For \( Y \in \mathbb{F}_{2^n}^* \), find the preimages in \( \mathbb{F}_{2^n} \) of \( Y \) under the Dickson polynomial \( D_{q+1} \), that is, find the elements of the set

\[
 D_{q+1}^{-1}(Y) = \{ z \in \mathbb{F}_{2^n}^* \mid D_{q+1}(z) = Y \}. \tag{6}
\]
On Problem 1: find $Y$ such that $a^{rac{g}{2}} = \frac{1}{f_{k,g+1}(\frac{1}{Y})}$

Recall:

**Proposition**

Let $n$ be a positive integer. Then, every element $z$ of $\mathbb{F}_{2^n}^*$ can be written (twice) $z = c + \frac{1}{c}$ where $c \in \mathbb{F}_{2^n}^* \cup M$ with $c \neq 1$ and where $M = \{ \zeta \in \mathbb{F}_{2^n} | \zeta^{2^n+1} = 1 \}$
On Problem 1: find $Y$ such that $\alpha^q = \frac{1}{f_{k,q+1}(\frac{1}{Y})}$.

One has $Y = T + \frac{1}{T}$ where $T \in \mathbb{F}_{2^n} \setminus \mathbb{F}_2$ or $T \in M \setminus \{1\}$ where $M = \{\zeta \in \mathbb{F}_{2^n}^{2n} \mid \zeta^{2^n+1} = 1\}$ (observe that $M \setminus \{1\} \subset \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^n}$). Now,

$$\frac{1}{Y} = \left(\frac{1}{T + 1}\right)^2 + \frac{1}{T + 1}.$$
On Problem 1: find \( Y \) such that 
\[
\alpha^2 = \frac{1}{f_{k,q+1}(\frac{1}{Y})}
\]

The next step is to use an approach used in [Dillon-Dobbertin 2004] by introducing 
\[
\Delta_k(X) = (X + 1)^{2^k-2^k+1} + X^{2^k-2^k+1} + 1
\]
and a permutation on \( \mathbb{F}_{2^n} \) defined as

\[
Q_{k,k'}(X) = \begin{cases} 
\frac{\sum_{i=1}^{k'} X^{2ik}}{X^{2k+1}} & \text{if } k' \text{ is odd} \\
\frac{\sum_{i=1}^{k'} X^{2ik} + 1}{X^{2k+1}} & \text{if } k' \text{ is even}
\end{cases}
\]

where \( k' \) is the inverse of \( k \) modulo \( n \), that is, \( kk' = 1 \mod n \). We then recall two properties of these polynomials [Dillon 1999]:

\[
\Delta_k(X) = \left( Q_{k,k'} \left( X + X^{2^k} \right) \right)^{-1} = f_{k,q+1}(X + X^2).
\]
On Problem 1: find $Y$ such that $a^\frac{q}{2} = \frac{1}{f_{k,q+1}(\frac{1}{Y})}$

Recall: $\Delta_k(X) = \left(Q_{k,k'} \left(X + X^{2^k}\right)\right)^{-1} = f_{k,q+1}(X + X^2)$ and $\frac{1}{Y} = \left(\frac{1}{T+1}\right)^2 + \frac{1}{T+1}$. Collecting together all the above discussion, we get

$$a^\frac{q}{2} = \left(f_{k,q+1} \left(\frac{1}{Y}\right)\right)^{-1} \iff a^{-\frac{q}{2}} = \Delta_k \left(\frac{1}{T+1}\right)$$

$$\iff a^{-\frac{q}{2}} = \frac{1}{Q_{k,k'} \left(\left(\frac{1}{T+1}\right)^q + \left(\frac{1}{T+1}\right)\right)}$$
On Problem 1: find \( Y \) such that \( a^\frac{q}{2} = \frac{1}{f_{k,q+1} \left( \frac{1}{Y} \right)} \)

**Proposition (K. H. Kim - SM 2019)**

Let \( a \in \mathbb{F}_{2^n}^* \). Let \( T \in \mathbb{F}_{2^n} \setminus \mathbb{F}_2 \cup M \setminus \{1\} \) be a solution of

\[
R_{k,k'} \left( a^{-\frac{q}{2}} \right) = \left( \frac{1}{T+1} \right)^q + \left( \frac{1}{T+1} \right)
\]

where \( R_{k,k'} \) is the compositional inverse of \( 1/Q_{k,k'} \). Then, \( Y = T + \frac{1}{T} \) is the unique solution of \( a^\frac{q}{2} = \left( f_{k,q+1} \left( \frac{1}{Y} \right) \right)^{-1} \).

the proposition above shows that solving Problem 1 amounts to find the solutions of a linear equation of the form \( x^q + x = b \). The polynomial expression of the solutions of such a linear equation has been given.
On Problem 1: find $Y$ such that $a^\frac{q}{2} = \frac{1}{f_{k,q+1}(\frac{1}{Y})}$

Define $S_{n,k}(x) = \sum_{i=0}^{n-1} x^{2^ki}$. Then,

**Proposition**

Let $\zeta$ be an element of $M \setminus \{1\}$. Then, for any $b \in \mathbb{F}_{2n}^*$, we have

$$\{x \in \mathbb{F}_{2n} \mid x + x^q = b\} = S_{n,k}(\frac{b}{\zeta + 1}) + \mathbb{F}_2$$
On Problem 1: find $Y$ such that $a^{\frac{q}{2}} = \frac{1}{f_{k,q+1}(\frac{1}{Y})}$

We can now explicit the solutions of Problem 1.

**Theorem**  (K. H. Kim - SM 2019)

Let $a \in \mathbb{F}^*_2$. Let $k'$ be the inverse of $k$ modulo $n$. Then, the unique solution of Problem 1 in $\mathbb{F}^*_2$ is $Y = T + \frac{1}{T}$ where

$$T = \frac{1}{S_{n,k} \left( \frac{R_{k,k'} \left( a^{-\frac{q}{2}} \right)}{\zeta+1} \right)} + 1$$

where $\zeta$ denotes any element of $\mathbb{F}_2^{2n}$ such that $\zeta^{2^n+1} = 1$, $S_{n,k}(x) = \sum_{i=0}^{n-1} x^{2^i}$ and $R_{k,k'}$ stands for the compositional inverse of $1/Q_{k,k'}$ defined by (7).

Furthermore, we have

$$Y = \frac{1}{S_{n,k} \left( \frac{R_{k,k'} \left( a^{-\frac{q}{2}} \right)}{\zeta+1} \right)} + \left( S_{n,k} \left( \frac{R_{k,k'} \left( a^{-\frac{q}{2}} \right)}{\zeta+1} \right) \right)^2$$
On Problem 1: find $Y$ such that $a^{q^2} = \frac{1}{f_{k,q+1}(\frac{1}{Y})}$

**Remark**

One can derive the polynomial representation of the inverse $R_{k,k'}$ of the mapping induced by $1/Q_{k,k'}$ on $\mathbb{F}_{2^n}$. This question has been studied in [Dillon-Dobbertin 2004] where it is introduced the following sequences of polynomials:

$$A_1(x) = x, \quad A_2(x) = x^{q+1}, \quad A_{i+2}(x) = x^{q^i} A_{i+1}(x) + x^{q^i+1-q^i} A_i(x), \quad i \geq 1,$$

$$B_1(x) = 0, \quad B_2(x) = x^{q-1}, \quad B_{i+2}(x) = x^{q^i} B_{i+1}(x) + x^{q^i+1-q^i} B_i(x), \quad i \geq 1.$$

The polynomial expression of $R_{k,k'}$ is then $R_{k,k'}(x) = \sum_{i=1}^{k'} A_i(x) + B_{k'}(x)$. 
On Problem 2 : find $D_{q+1}^{-1}(Y) = \{z \in \mathbb{F}_{2n}^* \mid D_{q+1}(z) = Y\}$

Write $z = c + \frac{1}{c}$ where $c \in \mathbb{F}_{2n}^*$ or $c \in M \setminus \{1\}$. One gets

$$Y = D_{q+1}(z) = c^{q+1} + \frac{1}{c^{q+1}} = T + \frac{1}{T}$$  \hfill (10)

with $T = c^{q+1}$

The equation $T + \frac{1}{T} = Y$ has two solutions in $\mathbb{F}_{2n}^* \cup M$ for any $Y \in \mathbb{F}_{2n}^*$ because it is equivalent to the quadratic equation $(\frac{T}{Y})^2 + \frac{T}{Y} = \frac{1}{Y^2}$ and that $Tr_1^{2n}(\frac{1}{Y}) = 0$ since $Y \in \mathbb{F}_{2n}$. In fact, we have two situations that occur depending on the value of $Tr_1^n(\frac{1}{Y})$:

- If $Tr_1^n(\frac{1}{Y}) = 0$, $T + \frac{1}{T} = Y$ has two solutions in $\mathbb{F}_{2n} \setminus \mathbb{F}_2$;

- If $Tr_1^n(\frac{1}{Y}) = 1$, $T + \frac{1}{T} = Y$ has two solutions in $M \setminus \{1\}$.

We shall now study separately those two cases.
On Problem 2 : find $D_{q+1}^{-1}(Y) = \{z \in \mathbb{F}_{2n}^* \mid D_{q+1}(z) = Y\}$

Suppose that $Tr_1^n (\frac{1}{Y}) = 0$. Denote $T$ and $\frac{1}{T}$ the two distinct elements of $\mathbb{F}_{2n} \setminus \mathbb{F}_2$ such that $T + \frac{1}{T} = Y$. Let us now turn our attention to the equation $c^{q+1} = T$ with $c \in \mathbb{F}_{2n}^* \cup M$, $c \neq 1$. Necessarily, $c \in \mathbb{F}_{2n}^*$ Recall that

$$(q + 1, 2^n - 1) = \begin{cases} 
1 & \text{if } n \text{ is odd} \\
3 & \text{if } n \text{ is even}
\end{cases}$$

Therefore, there are 0 or 3 elements $c$ in $\mathbb{F}_{2n} \setminus \mathbb{F}_2$ such that $c^{q+1} = T$ when $n$ is even while there is a unique $c$ when $n$ is odd.
On Problem 2: find \( D_{q+1}^{-1}(Y) = \{ z \in \mathbb{F}_{2n}^* \mid D_{q+1}(z) = Y \} \)

We can then conclude from the above discussion and calculation the following result.

**Theorem (K. H. Kim - SM 2019)**

Let \( Y \in \mathbb{F}_{2n}^* \) such that \( Tr^n_1 \left( \frac{1}{Y} \right) = 0 \). We have

1. If \( n \) is even, let \( T \) be any element of \( \mathbb{F}_{2n} \setminus \mathbb{F}_2 \) such that \( T + \frac{1}{T} = Y \). Then

\[
D_{q+1}^{-1}(Y) = \left\{ cw + \frac{1}{cw} \mid c^{q+1} = T, \ c \in \mathbb{F}_{2n} \setminus \mathbb{F}_2, \ w \in \mathbb{F}_4^* \right\}
\]

Notably, \( D_{q+1}^{-1}(Y) = \emptyset \) if there is no \( c \) in \( \mathbb{F}_{2n} \setminus \mathbb{F}_2 \) such that \( c^{q+1} = T \).

2. If \( n \) is odd, let \( T \) be any element of \( \mathbb{F}_{2n} \setminus \mathbb{F}_2 \) such that \( T + \frac{1}{T} = Y \). Then

\[
D_{q+1}^{-1}(Y) = \left\{ T \cdot \frac{1}{T^{q+1}} + \frac{1}{T^{q+1}} \right\}.
\]
On Problem 2: find $D_{q+1}^{-1}(Y) = \{ z \in \mathbb{F}_{2n}^* \mid D_{q+1}(z) = Y \}$

Next, suppose $Tr_1^n \left( \frac{1}{Y} \right) = 1$. In that case, the two elements $T$ and $\frac{1}{T}$ such that $T + \frac{1}{T} = Y$ are both in $M \setminus \{1\}$. Now,

$$(q + 1, 2^n + 1) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

**Theorem (K. H. Kim- SM 2019)**

Let $Y \in \mathbb{F}_{2n}^*$ such that $Tr_1^n \left( \frac{1}{Y} \right) = 1$. We have

1. **If** $n$ **is odd**, let $T$ be any element of $M \setminus \{1\}$ such that $T + \frac{1}{T} = Y$. Then

$$D_{q+1}^{-1}(Y) = \left\{ cw + \frac{1}{cw} \mid c^{q+1} = T, \ c \in \mathbb{F}_{2n} \setminus \mathbb{F}_2, \ w \in \mathbb{F}_{4}^* \right\}$$

   Notably, $D_{q+1}^{-1}(Y) = \emptyset$ if there is no $c$ in $\mathbb{F}_{2n} \setminus \mathbb{F}_2$ such that $c^{q+1} = T$.

2. **If** $n$ **is even**, let $T$ be any element of $M \setminus \{1\}$ such that $T + \frac{1}{T} = Y$. Then

$$D_{q+1}^{-1}(Y) = \left\{ T^{\frac{1}{q+1}} + \frac{1}{T^{\frac{1}{q+1}}} \right\}.$$
Solution of the equation (*) \( x^{2^k+1} + x + a = 0 \) in \( \mathbb{F}_{2^n} \) with \( (n, k) = 1 \)

Let \( k' \) be the inverse of \( k \) modulo \( n \). Let \( \zeta \in \mathbb{F}_{2^{2n}} \) such that \( \zeta \neq 1 \) and \( \zeta^{2^n+1} = 1 \). Define

\[
T = \frac{1}{S_{n,k} \left( \frac{R_{k,k'} \left( a - \frac{q}{2} \right)}{\zeta + 1} \right)} + 1.
\]

**Theorem (n is even (then k is necessarily odd)-K. H. Kim- SM 2019)**

1. If \( T \) is in \( \mathbb{F}_{2^n} \) but is not a cube of an element of \( \mathbb{F}_{2^n} \), Equation (*) has no solutions in \( \mathbb{F}_{2^n} \).

2. If \( T \) is in \( \mathbb{F}_{2^n} \) and is a cube of an element of \( \mathbb{F}_{2^n} \), Equation (*) has three distinct solutions in \( \mathbb{F}_{2^n} \) that can be written as \( \frac{\left( cw + \frac{1}{cw} \right)^{q-1}}{YT_k^q \left( \frac{1}{Y} \right)} \) where \( c^{q+1} = T \), \( w \in \mathbb{F}_4^* \) and \( Y = T + \frac{1}{T} \).

3. If \( T \) is in \( \mathbb{F}_{2^{2n}} \setminus \mathbb{F}_{2^n} \); Equation (*) has a unique solution in \( \mathbb{F}_{2^n} \) that can be written as \( \frac{\left( \frac{1}{T^q+1} + \frac{1}{T^{q+1}} \right)^{q-1}}{YT_k^q \left( \frac{1}{Y} \right)} \) where \( Y = T + \frac{1}{T} \).
Solution of the equation \((*)\) \(x^{2^k+1} + x + a = 0\) in \(\mathbb{F}_{2^n}\) with \((n, k) = 1\)

**Theorem** \((n \text{ is odd and } k \text{ odd- K. H. Kim- SM 2019})\)

Let \(M\) be the multiplicative subgroup of \(\mathbb{F}_{2^n}\) of order \(2^n + 1\). Then, we have:

1. If \(T\) is in \(M\) but is not a cube of an element of \(M\), the equation has no solutions in \(\mathbb{F}_{2^n}\).

2. If \(T\) is in \(M\) and is a cube of an element of \(M\), the equation has three distinct solutions in \(\mathbb{F}_{2^n}\) that can be written as \(\left(\frac{cw + \frac{1}{cw}}{2} \right)^{q-1} \frac{1}{YT_k^q \left(\frac{1}{Y}\right)}\) where \(c^{q+1} = T\), \(w \in \mathbb{F}_4^*\) and \(Y = T + \frac{1}{T}\).

3. If \(T\) is in \(\mathbb{F}_{2^n}\); the equation has a unique solution in \(\mathbb{F}_{2^n}\) that can be written as \(1 + \frac{1}{YT_k^q \left(\frac{1}{Y}\right)} \left(\frac{T \frac{1}{q+1} + \frac{1}{T}}{\frac{T^{q+1}}{2}}\right)^{q-1}\) where \(Y = T + \frac{1}{T}\).
Solution of the equation (*) \( x^{2^k+1} + x + a = 0 \) in \( \mathbb{F}_{2^n} \) with \((n, k) = 1\)

Let \( l = n - k \), \( q' = 2^l \) and \( l' \) the inverse of \( l \) modulo \( n \).

\[
T' = \frac{1}{S_{n,l} \left( R_{l,l'} \left( \frac{a - \frac{(q')^2}{2}}{\zeta + 1} \right) \right)} + 1.
\]

**Theorem (n odd, k even-K. H. Kim- SM 2019)**

Let \( M \) be the multiplicative subgroup of \( \mathbb{F}_{2^{2n}} \) of order \( 2^n + 1 \). Then, we have :

1. If \( T' \) is in \( M \) but is not a cube of \( M \), equation (*) has no solutions in \( \mathbb{F}_{2^n} \).
2. If \( T' \) is in \( M \) and is a cube of \( M \), equation (*) has three distinct solutions in \( \mathbb{F}_{2^n} : \left( \frac{dw + \frac{1}{dw}}{2} \right)^{q'-1} \right) \), where \( dq'+1 = T' \), \( w \in \mathbb{F}_{4}^* \) and \( Y' = T' + \frac{1}{T'} \).

\[
\left( T'_q \cdot \left( \frac{1}{T'} \right) + \frac{1}{T'_q \cdot 1} \right)^{q'-1}
\]

3. If \( T' \) is in \( \mathbb{F}_{2^n} \); equation (*) has one solution : \( 1 + \frac{\left( T'_q \cdot 1 + \frac{1}{T'_q} \right)}{\left( \frac{1}{T'} \right)} \), where \( Y' = T' + \frac{1}{T'} \).
Solution of the equation (*) $x^3 + x + a = 0$ in $\mathbb{F}_{2^n}$, $n$ even

Let $\zeta \in \mathbb{F}_{2^n}$ such that $\zeta \neq 1$ and $\zeta^{2^n+1} = 1$. Define

$$T = \frac{1}{S_{n,1}\left(\frac{a^{-1}}{\zeta+1}\right)} + 1.$$

where $S_{n,1}(x) = \sum_{i=0}^{n-1} x^{2^i}$.

1. If $T$ is in $\mathbb{F}_{2^n}$ but is not a cube of an element of $\mathbb{F}_{2^n}$, Equation (*) has no solutions in $\mathbb{F}_{2^n}$.

2. If $T$ is in $\mathbb{F}_{2^n}$ and is a cube of an element of $\mathbb{F}_{2^n}$, Equation (*) has three distinct solutions in $\mathbb{F}_{2^n}$ that can be written as $cw + \frac{1}{cw}$ where $c^3 = T$, $w \in \mathbb{F}_4^*$ and $Y = T + \frac{1}{T}$.

3. If $T$ is in $\mathbb{F}_{2^{2n}} \setminus \mathbb{F}_{2^n}$; Equation (*) has a unique solution in $\mathbb{F}_{2^n}$ that can be written as $T^{\frac{1}{3}} + \frac{1}{T^{\frac{1}{3}}}$ where $Y = T + \frac{1}{T}$. 
Solution of the equation (*) $x^3 + x + a = 0$ in $\mathbb{F}_{2^n}$, $n$ odd

Let $\zeta \in \mathbb{F}_{2^n}$ such that $\zeta \neq 1$ and $\zeta^{2^n+1} = 1$. Define

$$T = \frac{1}{S_{n,1} \left( \frac{a^{-1}}{\zeta+1} \right)} + 1.$$

where $S_{n,1}(x) = \sum_{i=0}^{n-1} x^{2^i}$.

Let $M$ be the multiplicative subgroup of $\mathbb{F}_{2^n}$ of order $2^n + 1$. Then, we have:

1. If $T$ is in $M$ but is not a cube of an element of $M$, the equation has no solutions in $\mathbb{F}_{2^n}$.

2. If $T$ is in $M$ and is a cube of an element of $M$, the equation has three distinct solutions in $\mathbb{F}_{2^n}$ that can be written as $cw + \frac{1}{cw}$ where $c^3 = T$, $w \in \mathbb{F}_4^*$ and $Y = T + \frac{1}{T}$.

3. If $T$ is in $\mathbb{F}_{2^n}$; the equation has a unique solution in $\mathbb{F}_{2^n}$ that can be written as $1 + T^{\frac{1}{3}} + \frac{1}{T}^{\frac{1}{3}}$ where $Y = T + \frac{1}{T}$. 

Partial results about the zeros of $P_a(x) = x^{2^k+1} + x + a$ in $\mathbb{F}_{2^n}$ have been obtained in [Bluher 2004], [Helleseth-Kholosha 2008],[Helleseth-Kholosha 2010] and [Bracken-Tan-Tan 2014].

- We provided explicit expression of all possible roots in $\mathbb{F}_{2^n}$ of $P_a(x)$ in terms of $a$ when $(n, k) = 1$.
- We showed that the problem of finding zeros in $\mathbb{F}_{2^n}$ of $P_a(x)$ in fact can be divided into two problems with odd $k$: to find the unique preimage of an element in $\mathbb{F}_{2^n}$ under a Müller-Cohen-Matthews (MCM) polynomial and to find preimages of an element in $\mathbb{F}_{2^n}$ under a Dickson polynomial. We completely solved these two independent problems.