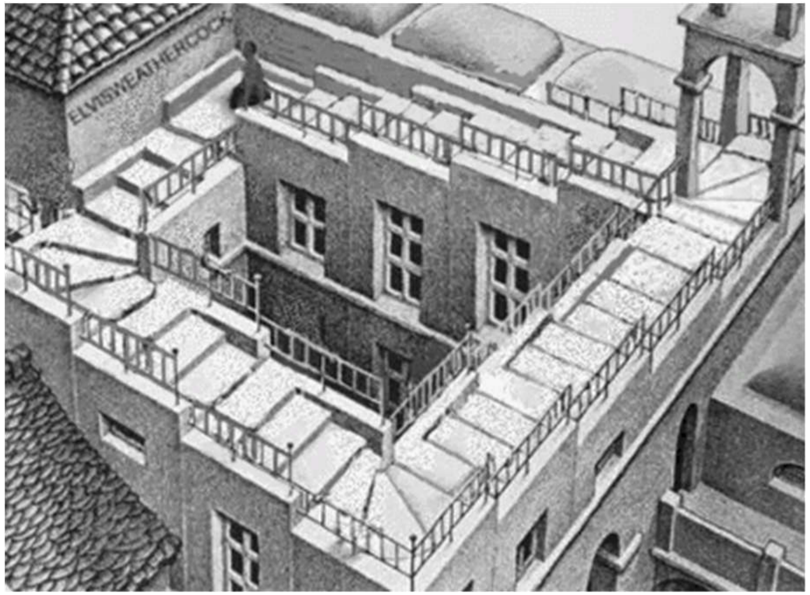
Boolean Functions and Resistance against NL Polynomial Invariant Attacks

[on Some Block Ciphers]



Nicolas T. Courtois

University College London, UK

Roadmap

eprint/2018/1242

- Non-Linear Cryptanalysis
  - Polynomial Invariants and Backdoors
- Can “strong” Boolean functions help to secure block ciphers against polynomial invariant attacks?
  - “product attack”
  - attacks based annihilators =>
    - potentially some attacks are HARD to avoid

Carlet Meta-Theorem:

"Almost all Boolean functions do not have any property we would wish them to have"

☹

- Claude Carlet: The complexity of Boolean functions from cryptographic viewpoint, Dagstuhl, 06111, 2006.
- Peter Clote, Evangelos Kranakis: Boolean functions, invariance groups, and parallel complexity, In SIAM J. Comput. 20 (3) pp. 553-590, 1991.

Partial Opposite [today]

Up to 15% of Boolean functions DO have the properties we need to make our NL attack work.

- Well, at least for some block ciphers…
- Proof of concept for T-310 for DES.

Question:

Why researchers have found
so few attacks on block ciphers?

LC = small HW words on 64 bits.

Question:

Why researchers have found
so few attacks on block ciphers?

"mystified by complexity"

lack of working examples: how a NL attack actually looks like??

Scope

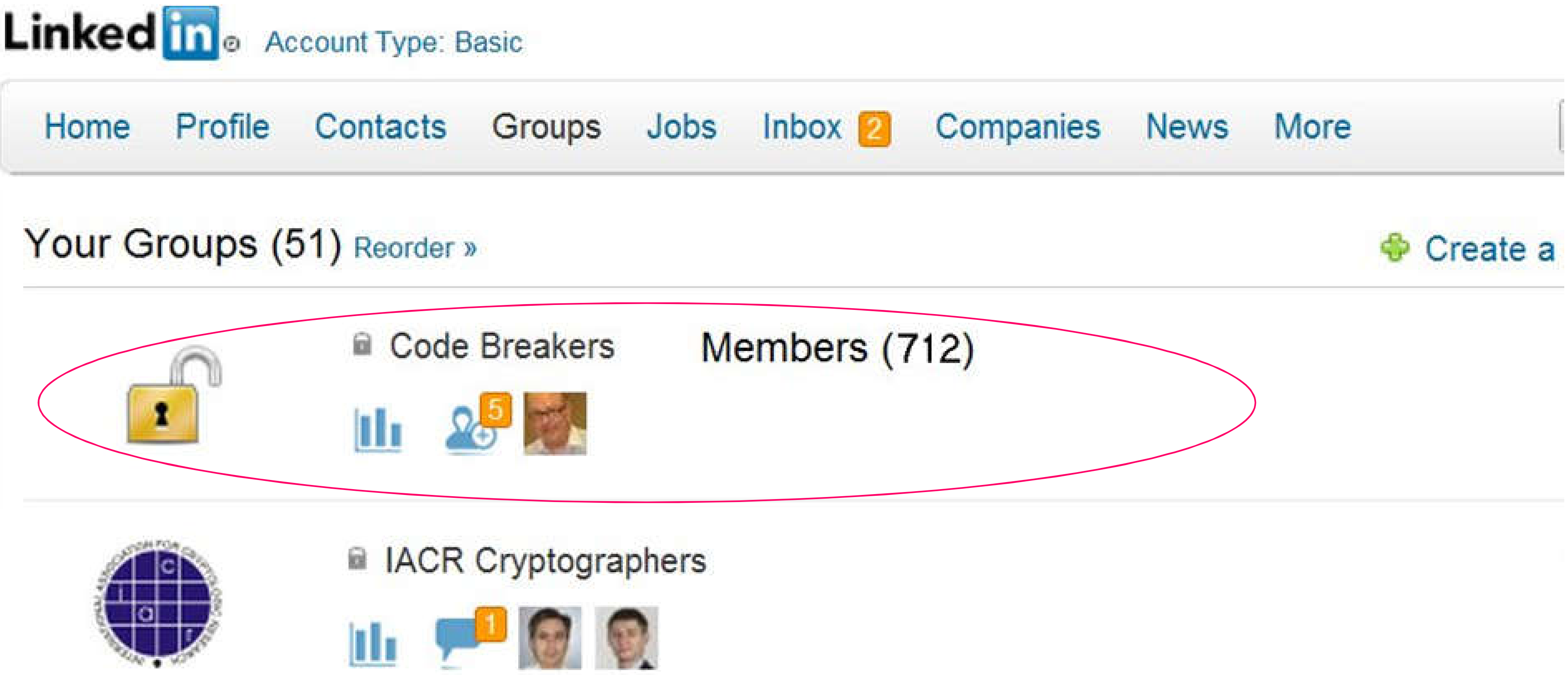
We study how an encryption function φ of a block cipher acts on arbitrary [Boolean] polynomials.

Stop, this is extremely complicated???

Claim:

Finding new attacks on block ciphers is EASY and FUN

Code Breakers - LinkedIn



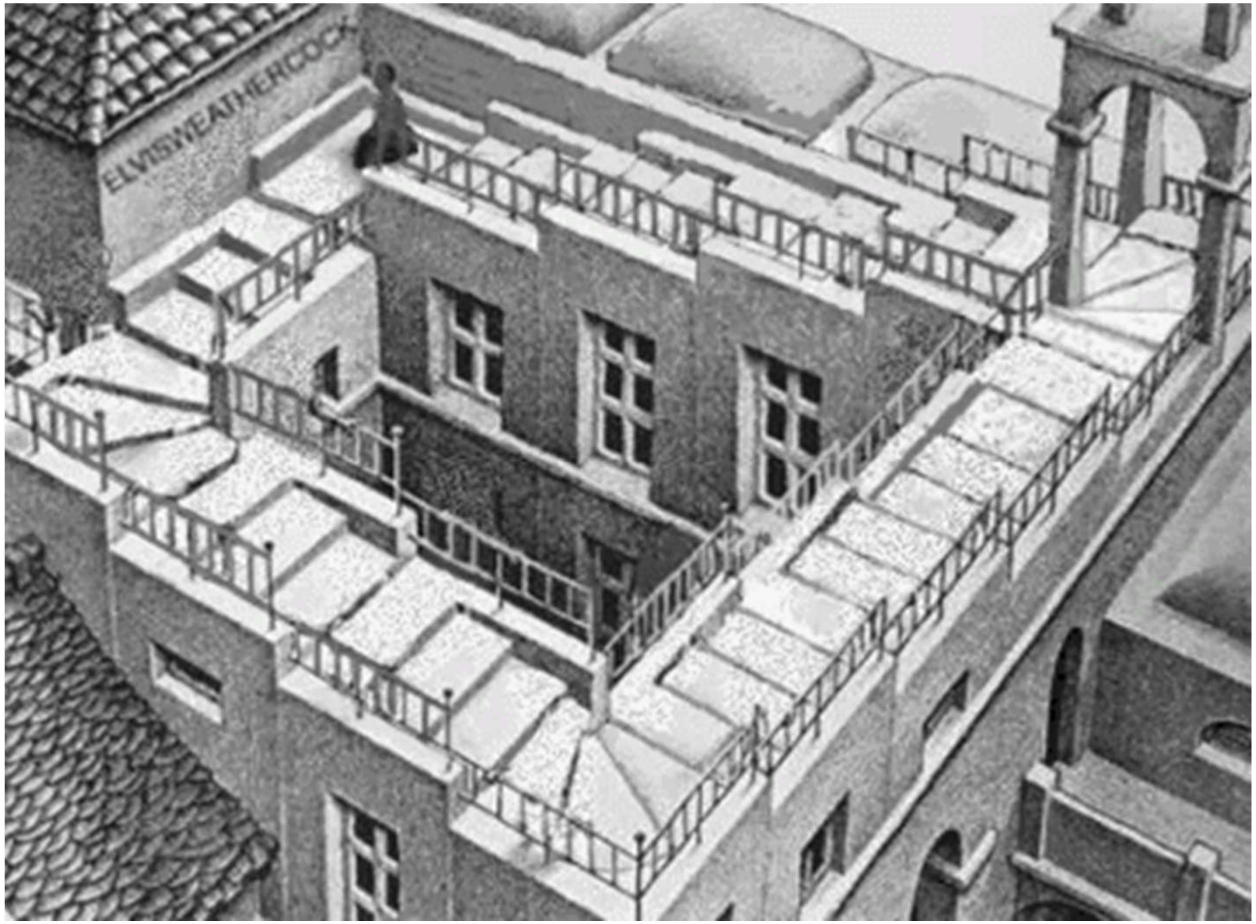
Cryptanalysis

=def=Making the impossible possible.

How? two very large polynomials with 16+ vars are simply equal

inspired by the master of impossible:

-- M. C. Escher



@eprint/2018/1242

Big Winner

"product attack"

a product of Boolean polynomials.

Claimed extremely powerful.
Why?

Definition

We say that

P => Q for 1R

if

P(inputs) = Q(outputs)

with proba =1, i.e. for every input

Another notation:

$$P = Q^{\varphi}$$

φ is 1 round of encryption

<=>

$P \Rightarrow Q$ for 1R

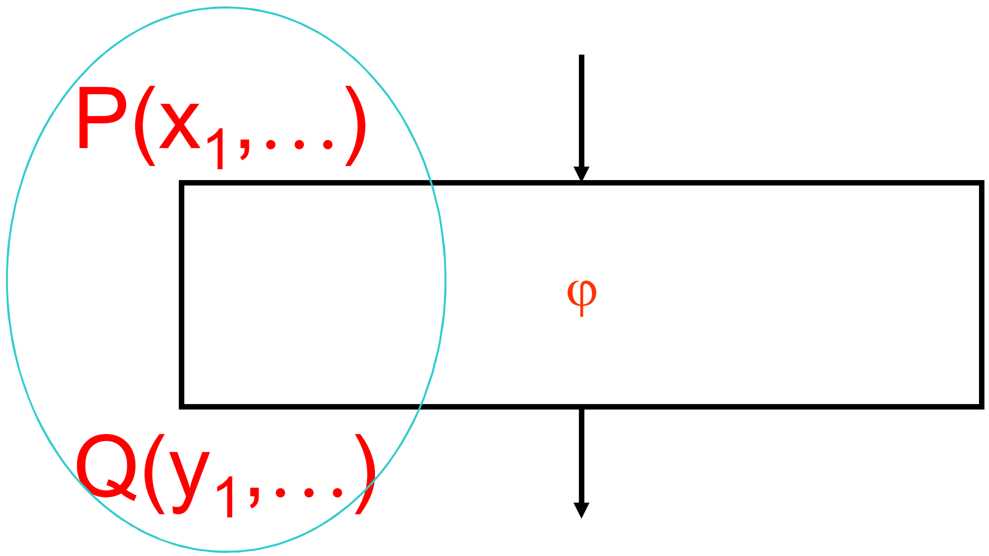
<=>

P(inputs) = Q(outputs)

for any input with Pr=1

Main Problem:

Two polynomials P => Q.



is P=Q possible??

“Invariant Theory” [Hilbert]: set of all invariants for <u>any</u> block cipher forms a [graded] finitely generated [polynomial] ring. A+B; A*B

Key Remark:

To insure that

P * R => P * R

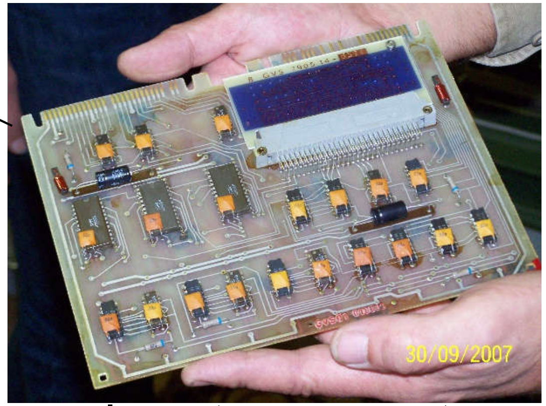
we only need to make sure that P=>P but ONLY for a subspace where R(inp)=1 and R(out)=1

East German T-310 Block Cipher

240 bits

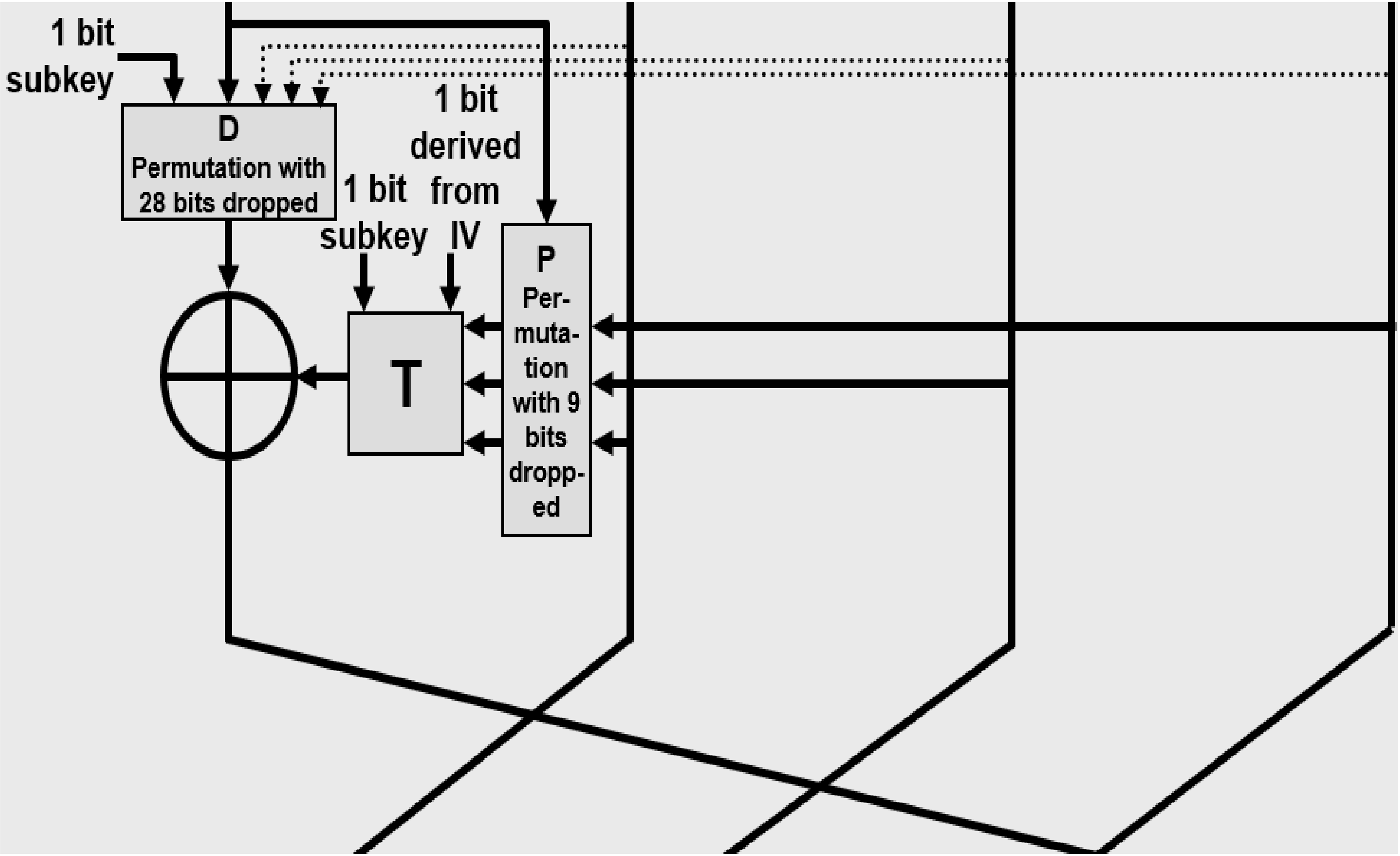
“quasi-absolute security”
[1973-1990]

has a physical RNG=>IV



long-term secret
90 bits only!

T-310 [1973-1990] – Feistel with 4 branches



blog.bettercrypto.com

How to Backdoor a Block Cipher

Posted by admin on 7 September 2018, 7:01 pm

I have written an elementary tutorial and a first proof of concept about how to backdoor a block cipher in a quite general setting. Potentially it applies to any block cipher.
Success is not guaranteed though, see the paper.

ADDED 2 JAN 2019:
a new paper shows that invariants of higher degree are substantially more powerful. Instead of a progression, we have a qualitative leap in what can be now achieved: see new paper.

ADDED 4 April 2019: here are slides presented at WCC 2019.

BTC Donate!

Filed under Code Breakers, Cryptanalysis, Crypto, Crypto History, Cybersecurity, Ethical Questions, Hall of Fame, Maths | Comment | Permalink

“Official” History of Cryptanalysis

- DC was known @IBM in 1970s

- Linear Cryptanalysis: Gilbert and Matsui [1992-93]

LC in 1976 [Eastern Germany]

Definition 3.1-1

$$\Delta_\alpha^g = 2^{n-1} - \| g(x) + (\alpha, x) \| \quad \forall \alpha \in \overline{0, 2^n - 1} .$$

$\|g\| \underset{\text{Def.}}{=} \sum_x g(x)$ $\qquad (\alpha, x) = \sum_{i=1}^{n} \alpha_i x_i$

Geheime Verschlußsache
MfS -020-Nr.: XI/493/76 Bl. 18

BSTU
0251

Ergebnisse :

Sei t die Anzahl der Übereinstimmungen der Funktionswerte von z.

Tabelle 3.1-2

α	Δ_α^z	t	α	Δ_α^z	t
0 0 0 0 0 0	~~32~~ 0	32	L 0 0 0 0 0	0	32
0 0 0 0 0 L	2	34	L 0 0 0 0 L	6	38
0 0 0 0 L 0	-4	28	L 0 0 0 L 0	0	32
0 0 0 0 L L	6	38	L 0 0 0 L L	6	38
0 0 0 L 0 0	-4	28	L 0 0 L 0 0	-4	28
0 0 0 L 0 L	-2	30	L 0 0 L 0 L	2	34
0 0 0 L L 0	0	32	L 0 0 L L 0	4	36
0 0 0 L L L	2	34	L 0 0 L L L	2	34

Generalised Linear Cryptanalysis = GLC =

[Harpes, Kramer and Massey, Eurocrypt'95]

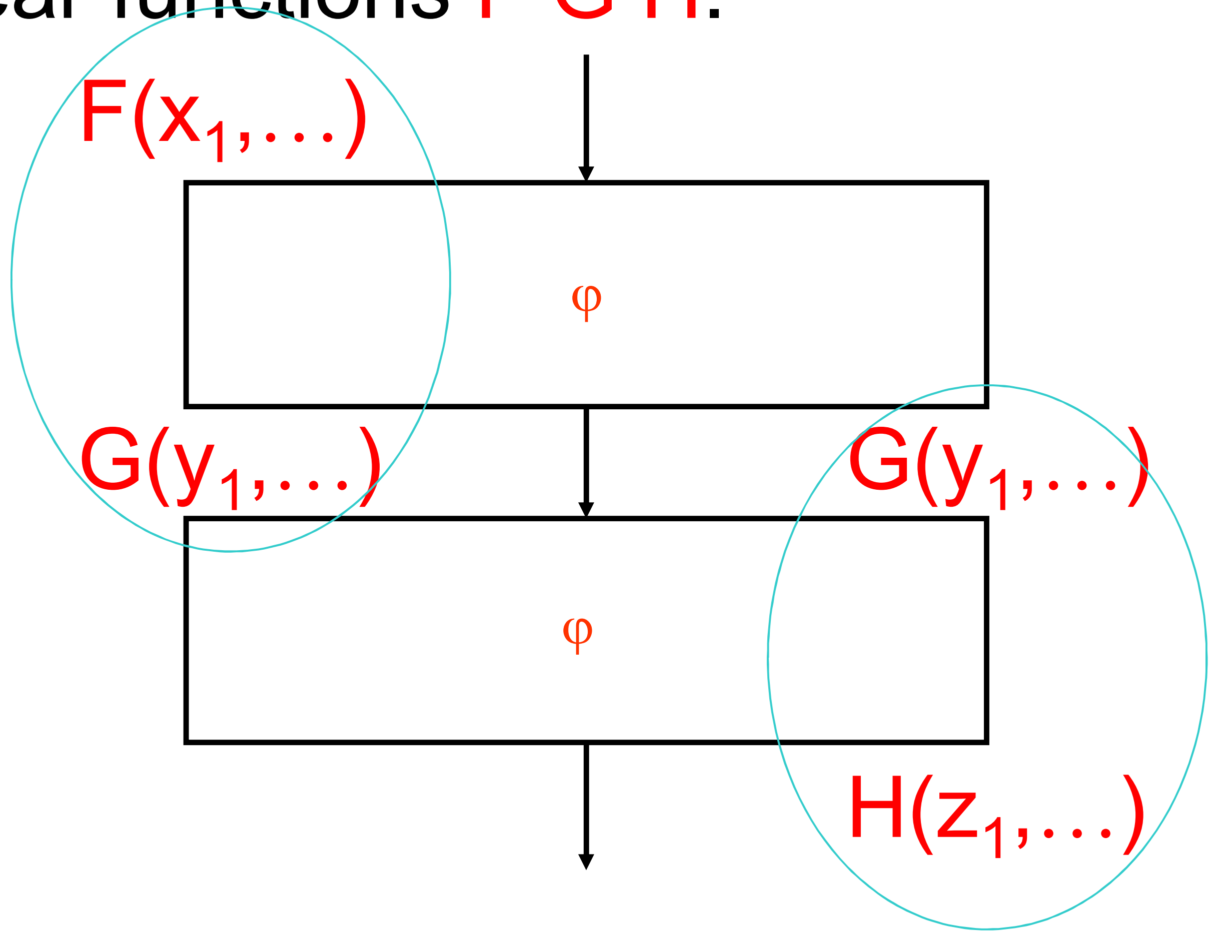
Concept of [invariant] non-linear I/O sums.

P(inputs) = P(outputs)
with some probability…

Connecting Non-Linear Approxs.

Black-Box Approach

Non-linear functions F G H.



GLC and Feistel Ciphers?

[Knudsen and Robshaw, EuroCrypt'96

"one-round approximations that are non-linear […] cannot be joined together"…

At Crypto 2004 Courtois shows that GLC is in fact possible for Feistel schemes!

BLC better than LC for DES

$$
\begin{aligned}
& L_0[3,8,14,25] \oplus L_0[3]R_0[16,17,20] \oplus R_0[17] \oplus \\
(*)\quad & L_{11}[3,8,14,25] \oplus L_{11}[3]R_{11}[16,17,20] \oplus R_{11}[17] = \\
& K[sth] + K[sth']L_0[3] + K[sth'']L_{11}[3]
\end{aligned}
$$

Better than the best existing linear attack of Matsui

for 3, 7, 11, 15, … rounds.

Ex: LC 11 rounds: $\frac{1}{2} \pm 1.91 \cdot 2^{-16}$

BLC 11 rounds: $\frac{1}{2} \pm 1.2 \cdot 2^{-15}$

Better Is Enemy of Good!

DES = Courtois @ Crypto 2004 :

$$\frac{1}{2} \pm 1.91 \cdot 2^{-16}$$ deg 1

$\downarrow$

$$\frac{1}{2} \pm 1.2 \cdot 2^{-15}$$ deg 2

$\downarrow$

proba=1.0 deg 10

New White Box Approach

non-linear I/O sums.

P(inputs) = P(outputs) with probability 1.

Formal equality of 2 polynomials.

Variable Boolean Function

We denote by Z our Boolean function

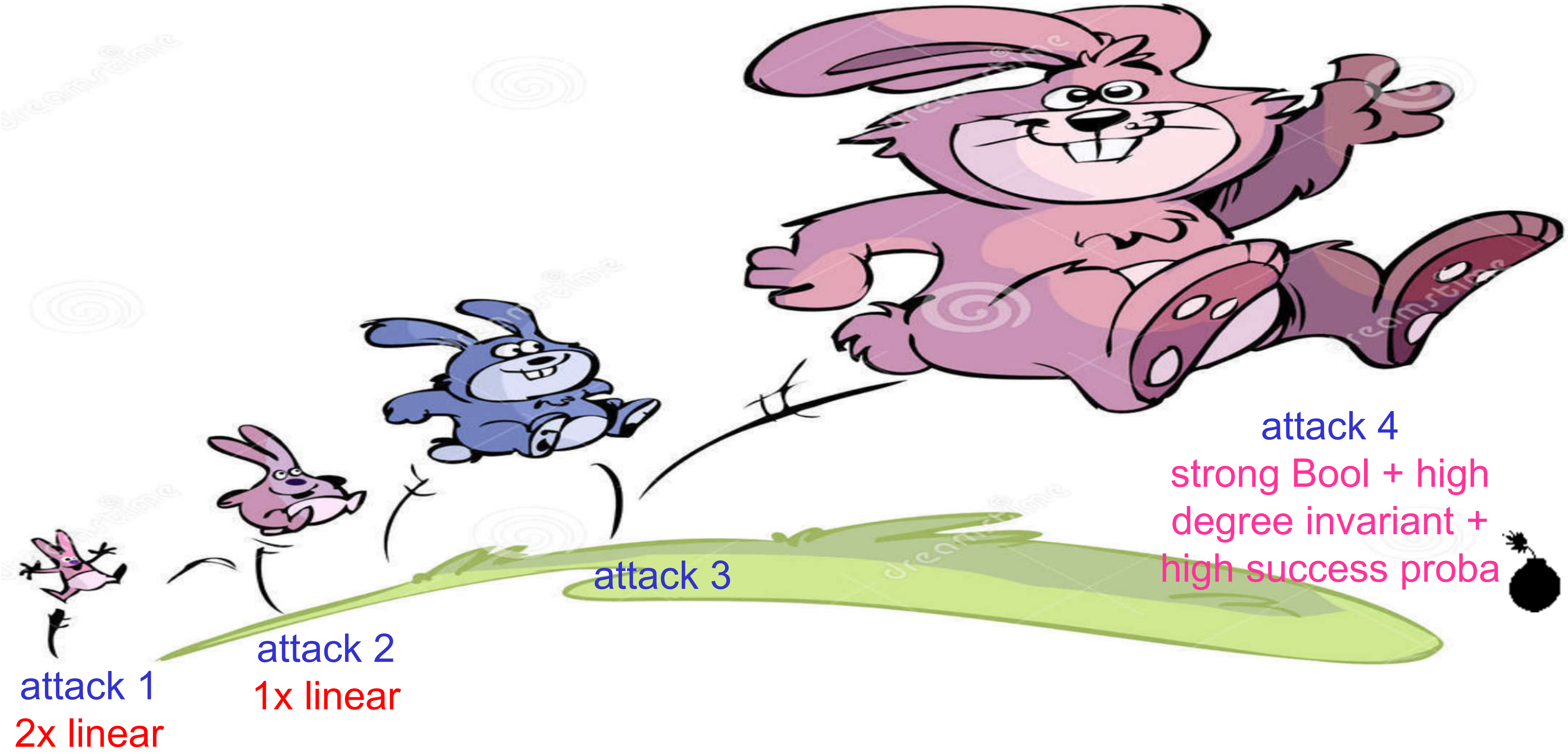
We consider a space of ciphers
where Z is variable.

Question: given a fixed polynomial P
what is the probability over random choice of Z that P(inputs) = P(outputs) is an invariant (for any number of rounds).

How Do You Find An Attack?

2^{2^n} possible attacks

Invariant Hopping



Group Theory – Is DES A Group?

Study of group generated by φ_K for any key K.

Typically AGL not GL. Any smaller sub-groups?

AN APPLICATION OF THE O'NAN-SCOTT THEOREM TO THE GROUP GENERATED BY THE ROUND FUNCTIONS OF AN AES-LIKE CIPHER

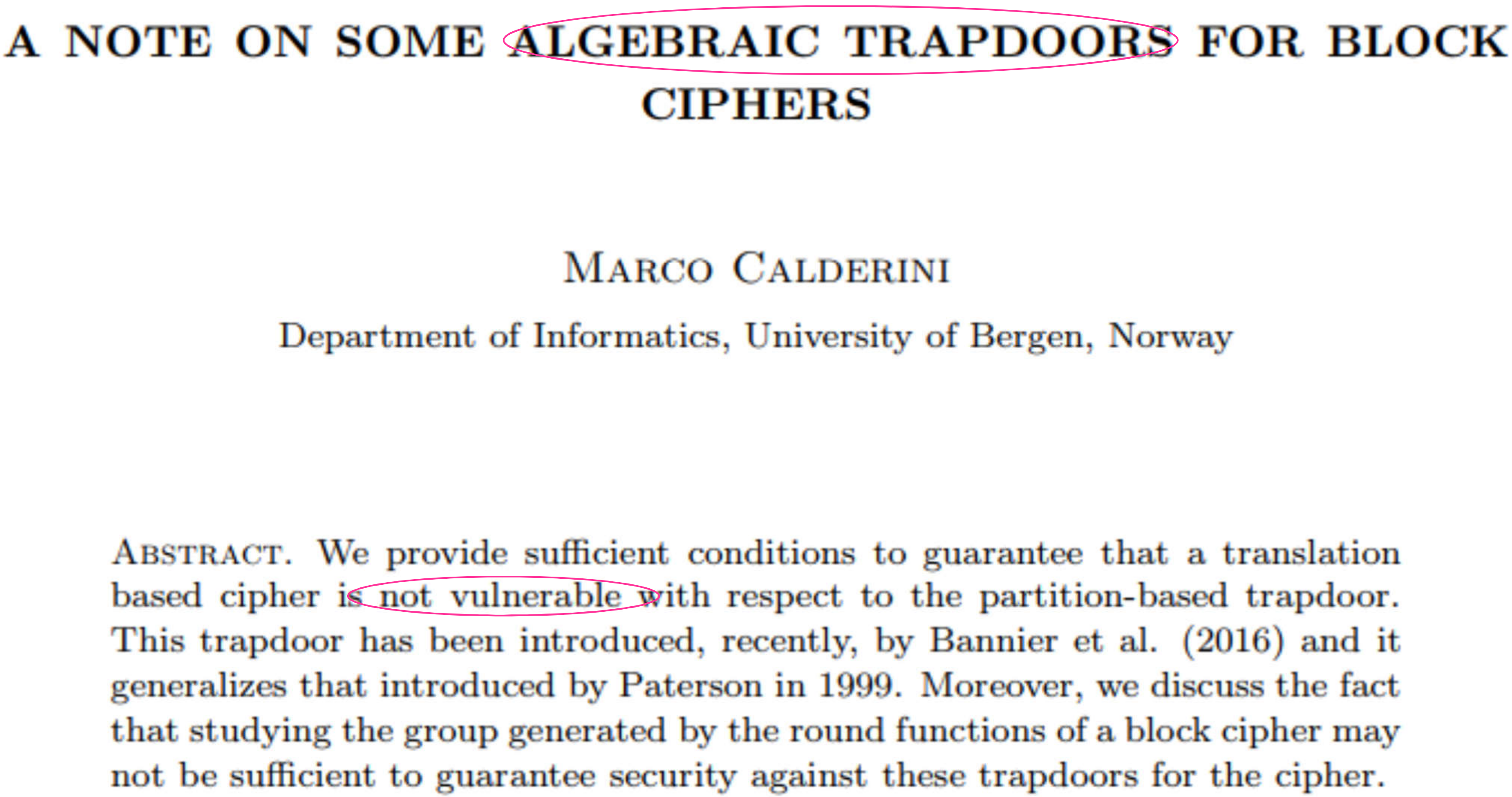
A. CARANTI, F. DALLA VOLTA, AND M. SALA

ABSTRACT. In a previous paper, we had proved that the permutation group generated by the round functions of an AES-like cipher is primitive. Here we apply the O'Nan Scott classification of primitive groups to prove that this group is the alternating group.

1. INTRODUCTION

According to Shannon [Sha49, p. 657], a cipher "is defined abstractly as a set of transformations". Coppersmith and Grossman [CG75], and later in 1988 Kaliski, Rivest and Sherman [KRS88], called attention to the group generated by a cipher.

Related Research


A NOTE ON SOME ALGEBRAIC TRAPDOORS FOR BLOCK CIPHERS

MARCO CALDERINI

Department of Informatics, University of Bergen, Norway

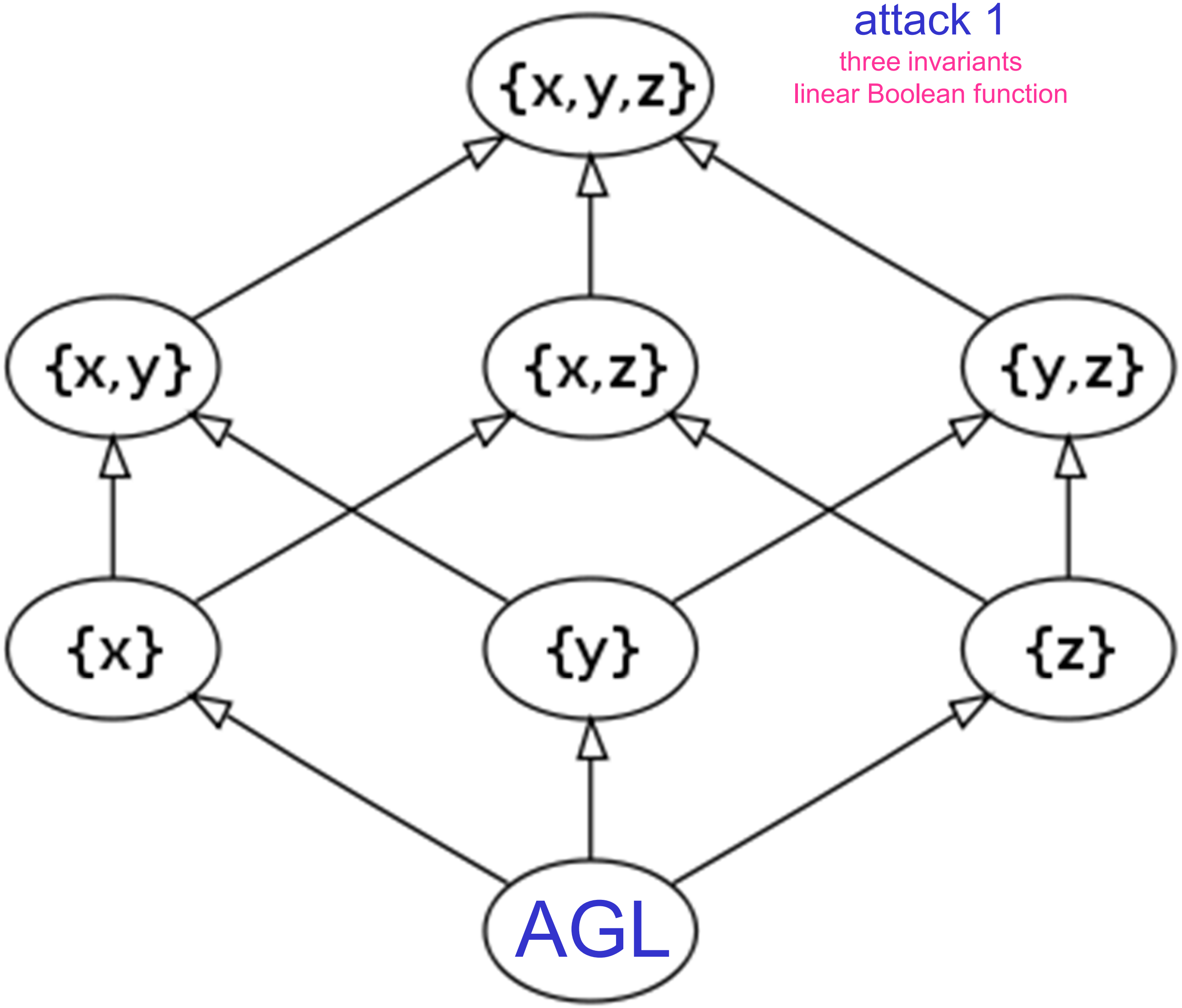
ABSTRACT. We provide sufficient conditions to guarantee that a translation based cipher is not vulnerable with respect to the partition-based trapdoor. This trapdoor has been introduced, recently, by Bannier et al. (2016) and it generalizes that introduced by Paterson in 1999. Moreover, we discuss the fact that studying the group generated by the round functions of a block cipher may not be sufficient to guarantee security against these trapdoors for the cipher.

Hopping in Group Lattices

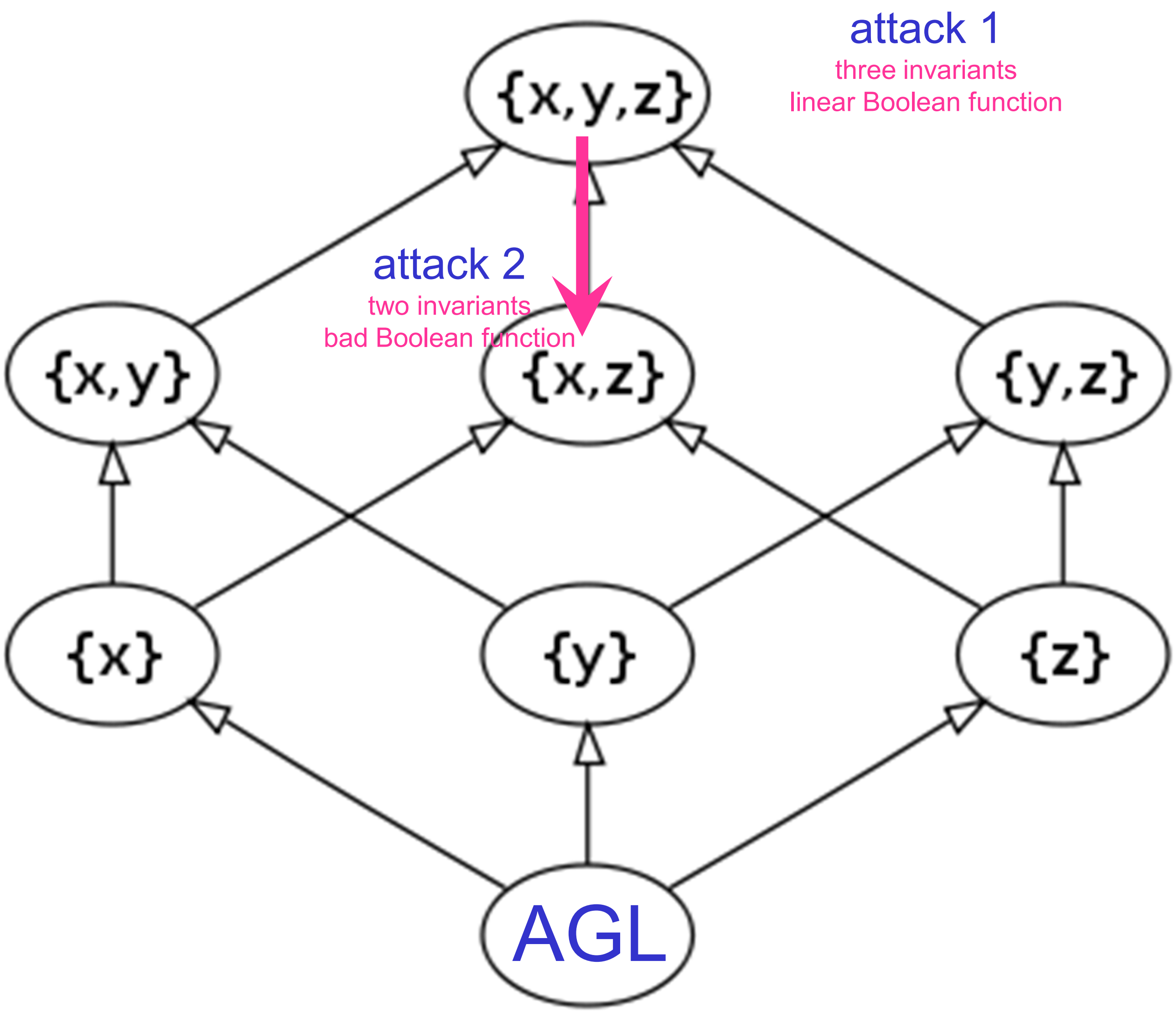
attack 1

three invariants

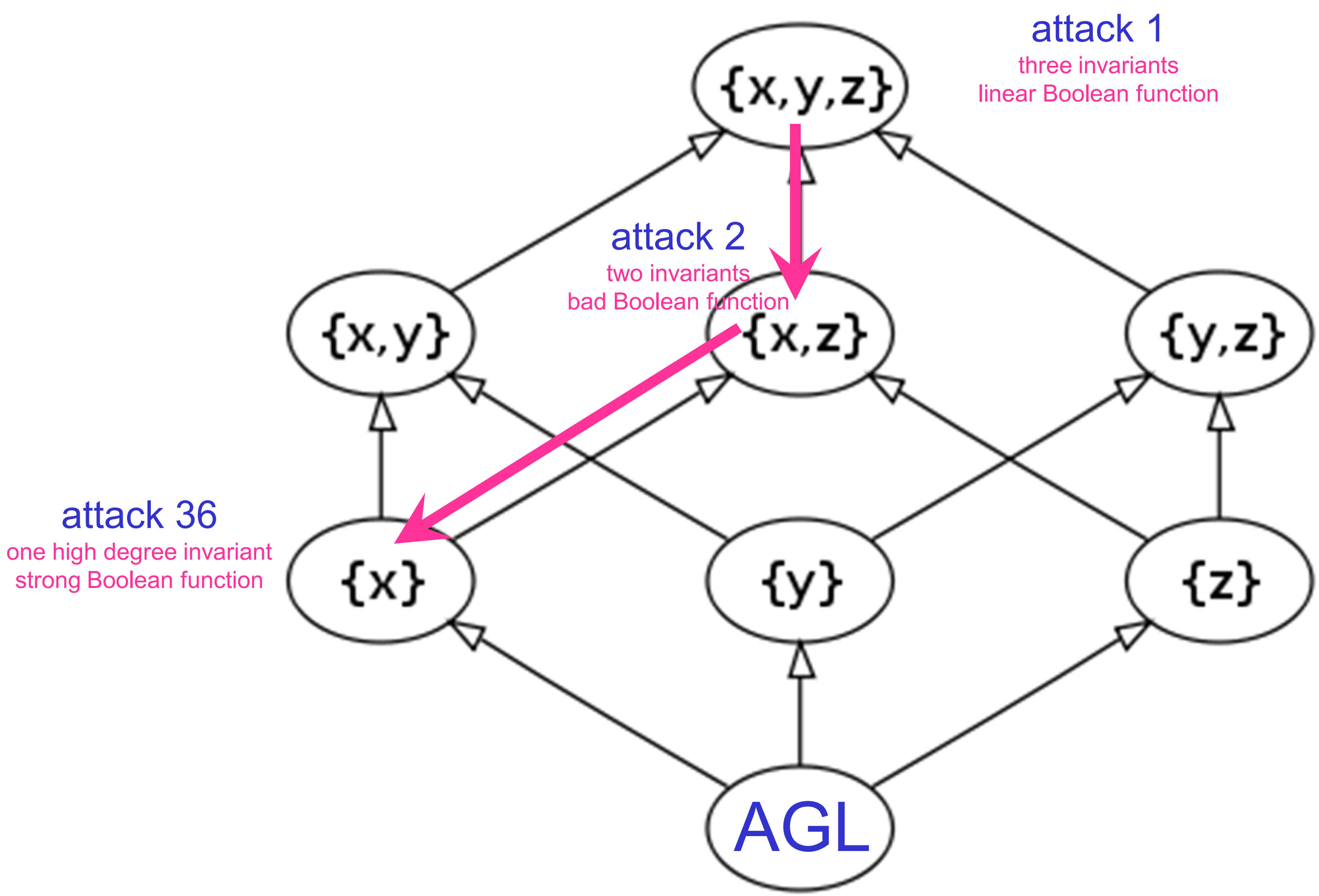
linear Boolean function



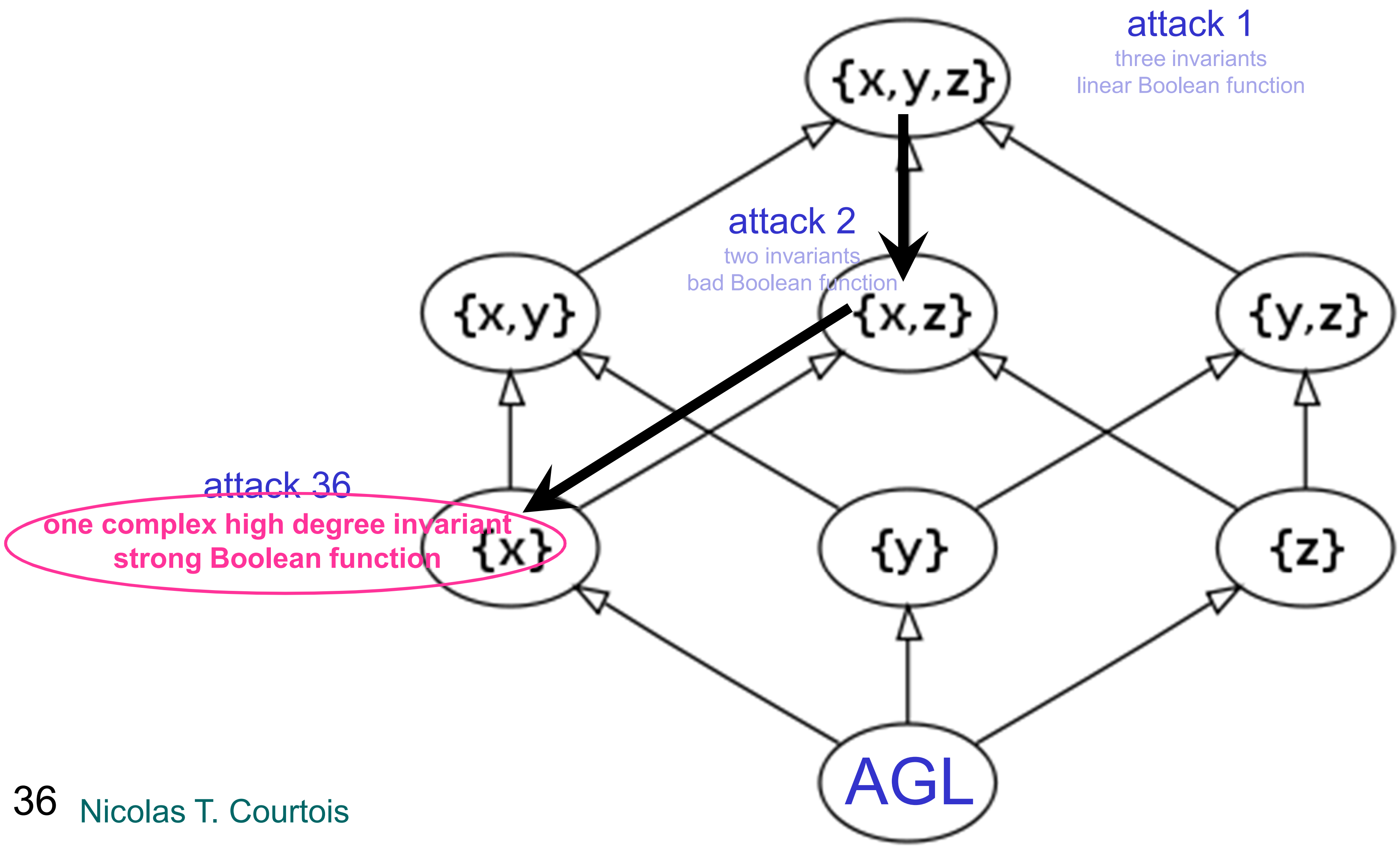
Hopping in Group Lattices



Hopping in Group Lattices



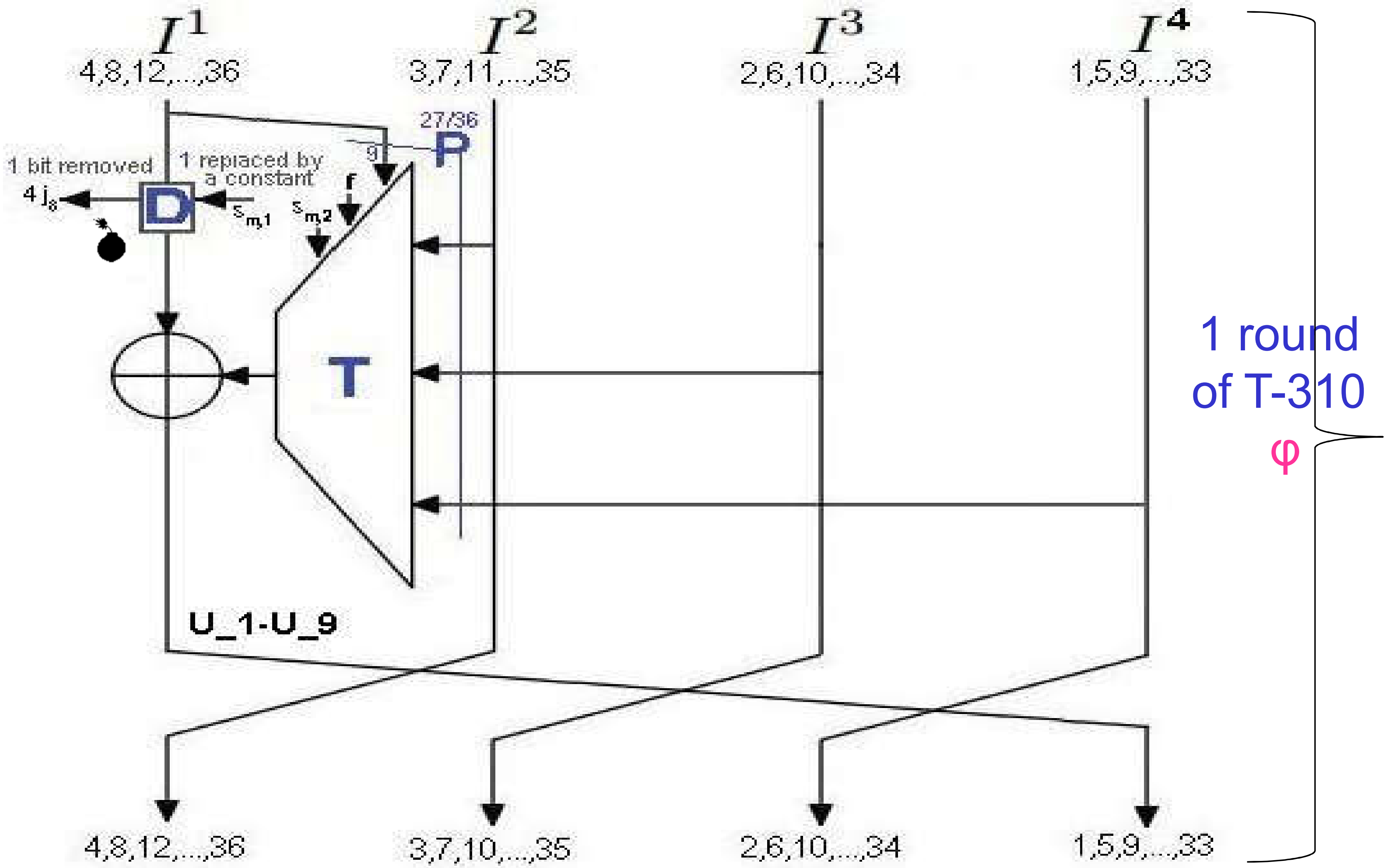
Hopping in Group Lattices



"Hopping" Discovery

- Learn from examples.
- Find a path from a trivial attack on a weak cipher to a non-trivial attack on a strong cipher.

T-310 [Contracting Feistel, 1970s, Eastern Germany!]



Impossible => Possible?

- We literally use “impossible” linear properties, which cannot happen and do not happen,

  and construct a non-linear attack which works.

Hopping Step 1 [WCC'19]

First we look at an attack where the Boolean function is linear and we have trivial LINEAR invariants (same as Matsui's LC)

Example:

$$Z(a,b,c,d,e,f) = f$$

$$\begin{cases} A \stackrel{def}{=} (e+m) \\ B \stackrel{def}{=} (f+n) \\ C \stackrel{def}{=} (g+o) \\ D \stackrel{def}{=} (h+p) \end{cases}$$

$$\text{4R linear invariant}$$

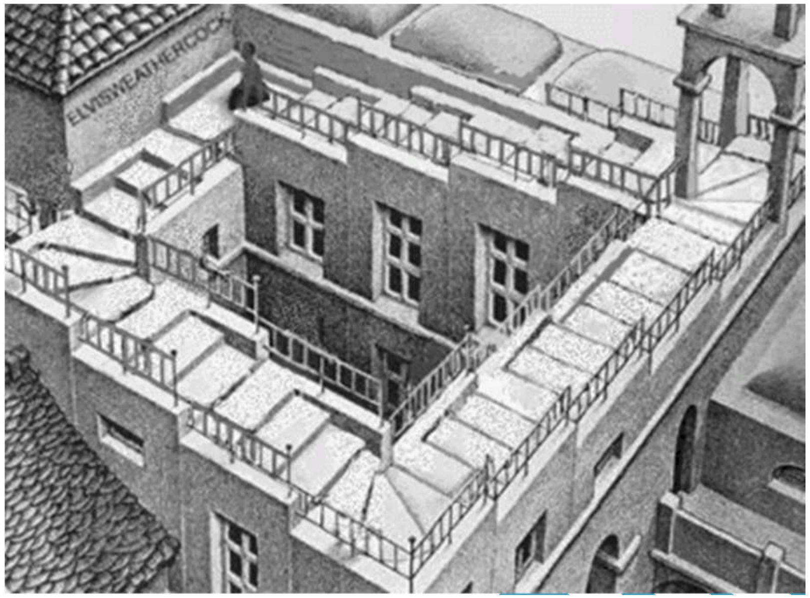
$$D \to C \to B \to A \overset{?}{\to} D$$

impossible transition

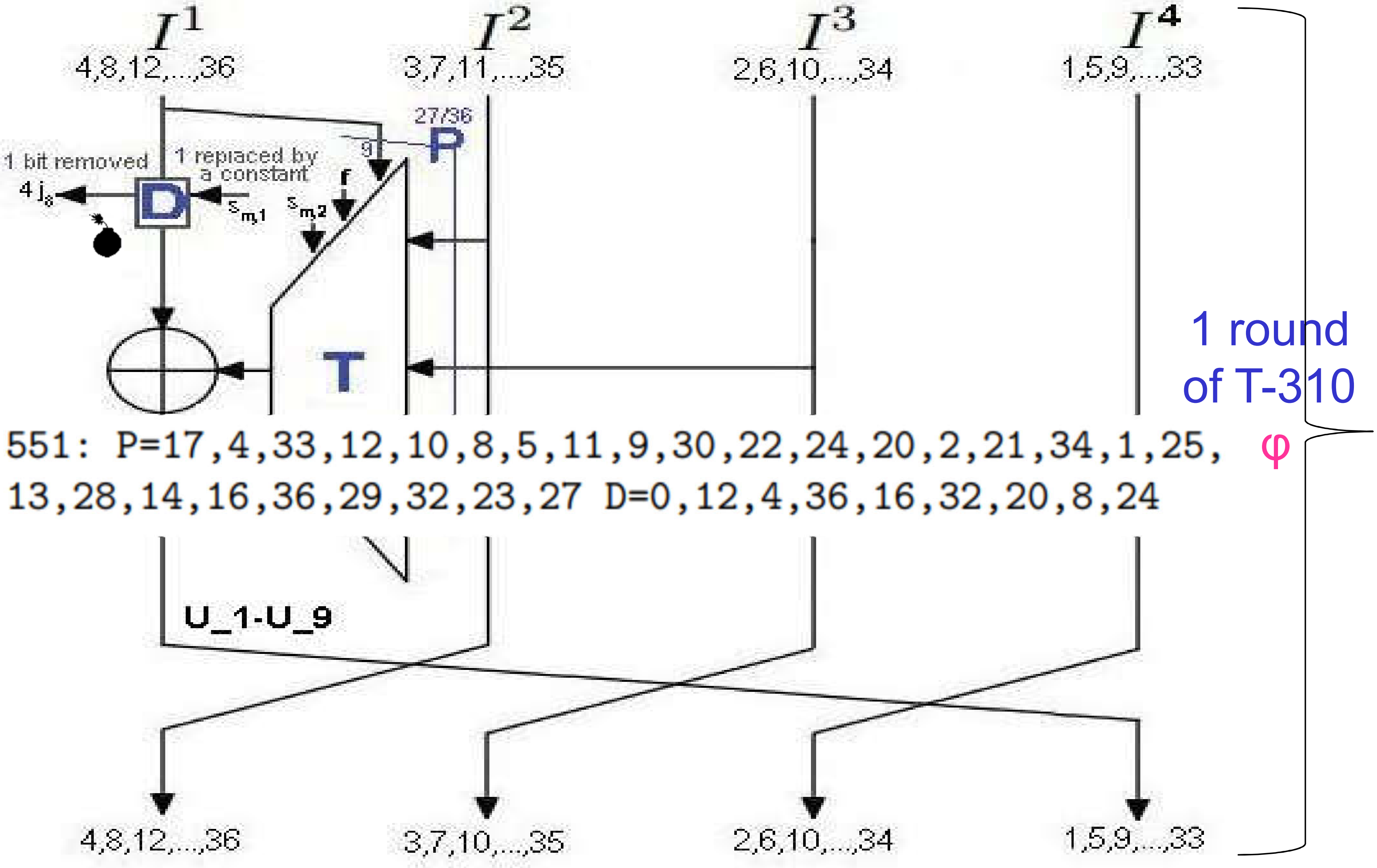
Impossible?

3 trivial, 1 impossible transitions

$$D \rightarrow C \rightarrow B \rightarrow A \overset{?}{\rightarrow} D$$



A Vulnerable Setup



551: P=17,4,33,12,10,8,5,11,9,30,22,24,20,2,21,34,1,25,
13,28,14,16,36,29,32,23,27 D=0,12,4,36,16,32,20,8,24

Hopping Step2 [WCC'19]

Now could you please tell us if

$$\mathcal{P} = eg + fh + eo + fp + gm + hn + mo + np$$

is an invariant?

=AC+BD

$$\begin{cases} A \stackrel{def}{=} (e+m) \\ B \stackrel{def}{=} (f+n) \\ C \stackrel{def}{=} (g+o) \\ D \stackrel{def}{=} (h+p) \end{cases}$$

Hopping Step2

Now could you please tell us if

$$\mathcal{P} = eg + fh + eo + fp + gm + hn + mo + np$$

is an invariant?

The answer is remarkably simple.

Hopping Step2

Theorem:

$$\mathcal{P} = eg + fh + eo + fp + gm + hn + mo + np$$

is an invariant

IF AND ONLY IF

a certain polynomial = FE =

Hopping Step2

Theorem:

$$\mathcal{P} = eg + fh + eo + fp + gm + hn + mo + np$$

is an invariant

IF AND ONLY IF

Fundamental
Equation

a certain polynomial = FE =

is zero (as a polynomial, multiple cancellations)

Compute FE?

Theorem:

$$\mathcal{P} = eg + fh + eo + fp + gm + hn + mo + np$$

is an invariant

IF AND ONLY IF

the Fundamental Equation FE

$$\mathcal{P}(a,b,c,d,e,f,g,h,\ldots,V) = \mathcal{P}(b,c,d,F+m,f,g,h,F+Z+O,$$

$$\ldots,F+Z+O+Y+q+L+X+i+W+j+K)$$

is zero

Compute FE?

Theorem:

$$\mathcal{P} = eg + fh + eo + fp + gm + hn + mo + np$$

is an invariant

IF AND ONLY IF

$$\underbrace{Y(g+o) = m(g+o)}$$

is zero (as a polynomial, multiple cancellations)

Notation

We have

$$\mathcal{P} = eg + fh + eo + fp + gm + hn + mo + np$$

is an invariant

IF AND ONLY IF

$$\mathcal{P} = \mathcal{P}(inputs) \stackrel{?}{=} \mathcal{P}(output\ \mathrm{ANF}) = \mathcal{P}^{\varphi}$$

IF AND ONLY IF

$$\underbrace{\mathcal{P} + \mathcal{P}^{\varphi}}_{} = \mathrm{FE}$$

is zero (as a polynomial, multiple cancellations)

Compact Notation

$\mathcal{P}$ is an invariant
IF AND ONLY IF

$$\underbrace{\mathcal{P} \stackrel{?}{=} \mathcal{P}\varphi}$$

= FE is zero

White Box Cryptanalysis = New

[Courtois 2018]

Same concept of a non-linear I/O sums.

Focus on perfect invariants mostly.

P(inputs) = P(outputs) with probability 1.

Formal equality of 2 polynomials.

Exploits the structure of the ring B_n.

- annihilation events ⇔ absorption events, nb. of vars collapses
- would be unthinkable if we had unique factorisation

ABCD=A'B'C'D'

New Paradigm [1905.04684]

https://arxiv.org/abs/1905.04684

Cornell University

arXiv.org > cs > arXiv:1905.04684

Computer Science > Cryptography and Security

Lack of Unique Factorization as a Tool in Block Cipher Cryptanalysis

Nicolas T. Courtois, Aidan Patrick

(Submitted on 12 May 2019)

Linear (or differential) cryptanalysis may seem dull topics for a mathematician: they are about super simple invariants characterized by say a word on n=64 bits with very few bits at 1, the space of possible attacks is small, and basic principles are trivial. In contract mathematics offers an infinitely rich world of possibilities. If so, why is that cryptographers have ever found so few attacks on block ciphers? In this paper we argue that black-box methods used so far to find attacks in symmetric cryptography are inadequate and we work with a more recent white-box algebraic methodology. Invariant attacks can be constructed explicitly through the study of roots of the so-called Fundamental Equation (FE). We also argue that certain properties of the ring of Boolean polynomials such as lack of unique factorization allow for a certain type of product construction attacks to flourish. As a proof of concept we show how to construct a complex and non-trivial attack where a polynomial of degree 7 is an invariant for any number of rounds for a complex block cipher.

Subjects: Cryptography and Security (cs.CR)
MSC classes: 13A50, 94A60, 68P25, 11T71, 14G50
ACM classes: E.3; I.1; K.2
Cite as: arXiv:1905.04684 [cs.CR]

$$\begin{cases} A \stackrel{def}{=} (e+m) \\ B \stackrel{def}{=} (f+n) \\ C \stackrel{def}{=} (g+o) \\ D \stackrel{def}{=} (h+p) \end{cases}$$

Conclusion Step2

Theorem:

$$\mathcal{P} = eg + fh + eo + fp + gm + hn + mo + np$$

is an invariant

IF AND ONLY IF

the Fundamental Equation FE

$$\underbrace{Y(g+o) = m(g+o)}$$

is zero (as a polynomial, multiple cancellations)

$$\begin{cases} A \stackrel{def}{=} (e+m) \\ B \stackrel{def}{=} (f+n) \\ C \stackrel{def}{=} (g+o) \\ D \stackrel{def}{=} (h+p) \end{cases}$$

What is Special About $\mathscr{P}$

2-factoring decomposition

$$\mathcal{P} = eg + fh + eo + fp + gm + hn + mo + np$$

= AC+BD.

is invariant IF AND ONLY IF

$$\underbrace{Y(g+o) = m(g+o)}$$

some solutions are:

$$Z(a,b,c,d,e,f) = f$$

$$Z = 1 + d + e + f + de + cde + def.$$

Attack of Degree 4

Q : Can we now have ABCD
to be an invariant of degree 4

Answer: <u>easy:</u> Y must be a root of

mBCD=YBCD

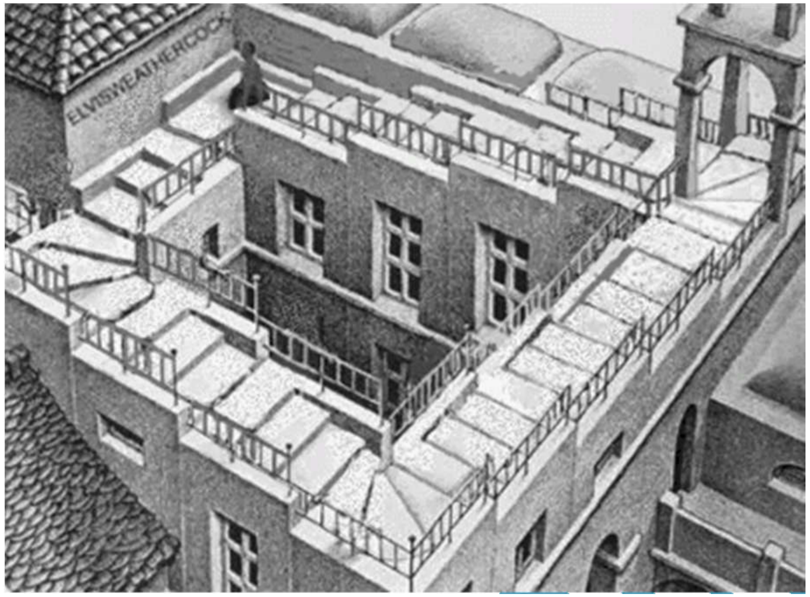
= FE

Product Attack

Construct NL invariants based on LC cycles:

$$A \to B \to C \to D \not\to A$$

Then ABCD is a round invariant of degree 4.



Phase Transition

When $\mathscr{P}$ is of degree 4, the Boolean function is still "inevitably" degenerated [WCC'18].

Q: Can we backdoor or break a cipher with a random Boolean function?

Solution:
The degree of $\mathscr{P}$ must increase to 8.

Phase Transition

When $\mathscr{P}$ is of degree 4, the Boolean function is still "inevitably" degenerated [this paper].

Q: Can we backdoor or break a cipher with a "strong" (e.g. random) Boolean function?

YES, see [eprint/2018/1242]

Degree 8 attack, $\mathscr{P}$=ABCDEFGH.

Thm 5.5.

In eprint/2018/1242 page 18.

$\mathcal{P}$=ABCDEFGH

is invariant if and only if

this polynomial vanishes:

$$FE = BCDFGH \cdot ((Y + E)W(.) + AY(.))$$

Can a polynomial with 16 variables with 2 very complex Boolean functions just disappear?

Hard Becomes Easy

Phase transition: eprint/2018/1242.

- When $\mathcal{P}$ degree grows, attacks become a LOT easier.
- Degree 8: extremely strong:

15% success rate over the choice of a random Boolean function and with $\mathcal{P}$=ABCDEFGH.

(3 variants)

WHAT??????????

$$\begin{cases} A \stackrel{def}{=} (e+m) \\ B \stackrel{def}{=} (f+n) \\ C \stackrel{def}{=} (g+o) \\ D \stackrel{def}{=} (h+p) \end{cases}$$

Let Y = Random Bool.

Can we HOPE that for
we have for example:

mBCD=YBCD i.e.

0=(Y+m)BCD = FE

Thm 6.0.1: Courtois-Meier Eurocypt 2003.

For any Z with 6 variables, Z or Z+1 always has some cubic annihilators.

Thm 6.4: [eprint/2018/1242]
For Z(a+b)(c+d)(e+f)=0, any Boolean function works with probability of 5%.

Less Trivial Attacks

an irregular sporadic attack with $\mathscr{P}$ of degree 7

Theorem 6.1 (A Degree 7 Invariant Attack). Let

$$\mathcal{P} = (A+B)\ (C+D)\ (D+F)(B+F)\ (E+F)(G+F)(G+H)$$

then $\mathcal{P}$ is a non-zero polynomial of degree 7. We also assume that

$$\begin{cases} \{D(2), D(3)\} = \{6\cdot 4, 7\cdot 4\} \\ \{D(6), D(7)\} = \{2\cdot 4, 3\cdot 4\} \end{cases}$$

and that inputs of Y are in order bits $27, 6, 10, 23, 21, 25$ and inputs of W are in order bits $26, 9, 5, 22, 7, 11$. We assume that the Boolean function used inside the cipher has after adding 1 TWO degree 3 annihilators as follows:

```
(Z+1)*(f+e)(d+a)(b+c)=0
(Z+1)*(f+e+1)(d+a+1)(b+c+1)=0
```

Then $\mathcal{P}$ is a round invariant for any key any IV and any number of rounds.

DES

problem:

a LOT more key bits

48 instead of 2 in each round

What about the actual DES wiring?

reality is more interesting than fiction!

Degree 5 Attack on DES

Theorem: Let $\mathscr{P}$=

$$(1+L06+L07)*L12 \ * \ R13*R24*R28$$

IF

(1+c+d)*W2==0 and (1+c+d)*X2==0

e*W3==0 and f*Z3==0

ae*X7==0 and ae*Z7==0

THEN

$\mathscr{P}$ is an invariant for 1 round of DES.