Resemblance

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Joint work

This is joint work with

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The work presented here comes from her Ph.D dissertation, which she completed recently with me at the University of Delaware.

Disclaimers

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The utility of the basic idea is up for debate – I think it could be important, but we haven't yet got something astounding from it.

Some notation

Throughout $\mathcal{G}$ denotes a finite group of order q, written additively but not necessarily abelian, and $\mathcal{G}^\star = \mathcal{G} \setminus \{0\}$.

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For a finite set $\mathcal{S}$, $\#\mathcal{S}$ denotes the cardinality of $\mathcal{S}$.

Now let $f : \mathcal{G} \to \mathcal{G}$.

Then $\mathrm{Im}(f) = \{f(x) : x \in \mathcal{G}\}$ denotes the image set of f,

and $V(f) = \#\,\mathrm{Im}(f)$ denotes the cardinality of the image set.

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The standard answer is $V(f)$, the size of the image set of f.
(Or $\#\mathcal{G} + 1 - V(f)$, if you're that way inclined.)

The motivating example

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Choose any $a \in \mathcal{G} \setminus \mathcal{H}$, so that $\mathcal{H}$ and $a + \mathcal{H}$ are the two cosets of $\mathcal{H}$ that partition $\mathcal{G}$.

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Question: Do you think f is nearly a permutation?

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Now define $g : \mathcal{G} \to \mathcal{G}$ by $g(h) = 0$ and $g(a + h) = a$ for all $h \in \mathcal{H}$.

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Question: Do you think f is nearly a permutation now?

The basic idea

Definition

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The *resemblance* $\mathrm{Res}(f, h)$ of f to h is defined by

$$\mathrm{Res}(f, h) = V(f - h).$$

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- $\mathrm{Res}(f, c + h) = \mathrm{Res}(f, h)$ for any constant $c \in \mathcal{G}$.
- $1 \leq \mathrm{Res}(f, h) \leq \#\mathcal{G}$.
  The minimum is achieved when $f = h$, while the maximum can be achieved when one of f or h is a constant and the other a bijection.

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The utility of this idea is in its application in certain directions.

For a given function $f : \mathcal{G} \to \mathcal{G}$, consider

$$\min\{\mathrm{Res}(f, h) : h \text{ has property P}\}.$$

This is a way to measure how far f is from having property P.

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We do at least have the extremes doing what we would want, for we have:

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And no, $\text{P-Res}(f) \neq \#\mathcal{G} + 1 - V(f)$. (At least not always!)

Intuition behind the idea

$$\text{P-Res}(f) = \min\{V(g) \,:\, g + f \in \Omega_{\mathcal{G}}\}.$$

Permutation resemblance is equal to the minimum value $V(g)$ (the smallest image size) as g runs through all functions on $\mathcal{G}$ for which $g + f$ is a permutation.

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It isn't hard to see that this is very different from $V(f)$; just think back to that example.

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So $g + f \in \Omega_{\mathcal{G}}$, but we can make n, and hence $V(f)$, arbitrarily large in this example, while the permutation resemblance will always be 2.

Better bounds

Let $f : \mathcal{G} \to \mathcal{G}$. We define two terms.

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For any $\text{PreIm}(f, b)$ of cardinality at least 2, choose distinct $x, y \in \text{PreIm}(f, b)$.

When $g + f$ is a permutation, we have

$$\begin{aligned}(g+f)(x) \neq (g+f)(y) &\Rightarrow g(x) + b \neq g(y) + b \\ &\Rightarrow g(x) \neq g(y),\end{aligned}$$

so that g must be injective on every preimage set of f, implying $u(f) \leq V(g)$ whenever $g + f$ is a permutation.

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The proof is by construction (of g).

Let g map exactly one element from each non-empty set $\text{PreIm}(f, b)$ to 0.

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At this point, both the domain and codomain of g have $\#\mathcal{G} - V(f)$ elements left unassigned.

Now pair off the unassigned domain/codomain elements (x, y) and set $g(x) = y - f(x)$.

This ensures $g + f \in \Omega_{\mathcal{G}}$ and $V(g) \leq \#\mathcal{G} + 1 - V(f)$ at worst.

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But the real question is how does P-Res really behave?

For starters, perhaps we should determine if it can be equal to either bound when they are not the same?

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The proof of that upper bound is on a worst-case scenario, so we don't expect that most functions will be at or near it, so we concentrated on the lower bound.

Two classes of functions that achieve the lower bound, I

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Theorem

Let $\eta(X) = X^{(p-1)/2} \in \mathbb{F}_p[X]$ with p an odd prime. Then

$$\text{P-Res}(\eta) = \begin{cases} u(\eta) + 1 = \frac{p+1}{2}, & \text{if } p \equiv 1 \pmod 4; \\ u(\eta) = \frac{p-1}{2}, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

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Note that $\eta(X)$ is the quadratic character over $\mathbb{F}_p$, so that $V(\eta) = 3$.
Thus we see

$$\frac{p \pm 1}{2} = \text{P-Res}(\eta) < p - 2$$

provided $p \geq 7$.

Indeed, we see P-Res is roughly half of the possible upper bound for this class of functions.

Two classes of functions that achieve the lower bound, II

Now let $q = p^e$ for some natural number e.

Recall that the set of p-polynomials over $\mathbb{F}_q$, that is those of the form

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represents the set of all linear operators of $(\mathbb{F}_q, +)$ when viewed as a vector space over $\mathbb{F}_p$.

Theorem

Any p-polynomial $L \in \mathbb{F}_q[X]$ satisfies P-Res$(L) = u(L)$.

Proof that linear operators meet the lower bound

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Further, for any $b \in \mathrm{Im}(L)$, $\mathrm{PreIm}(L, b) = \mathcal{N} + z$ where $L(z) = b$.

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All of this allows for a nice little argument.

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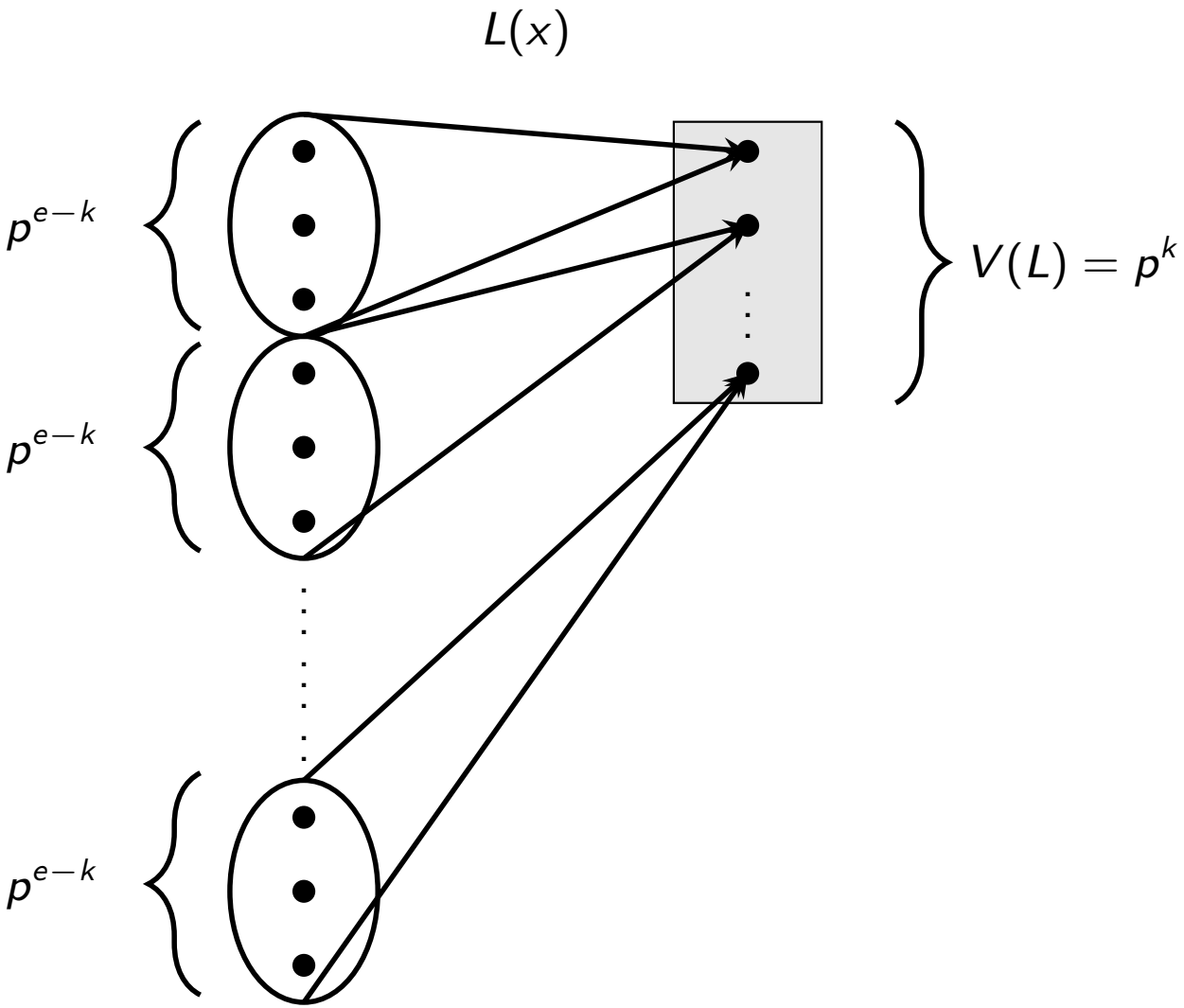

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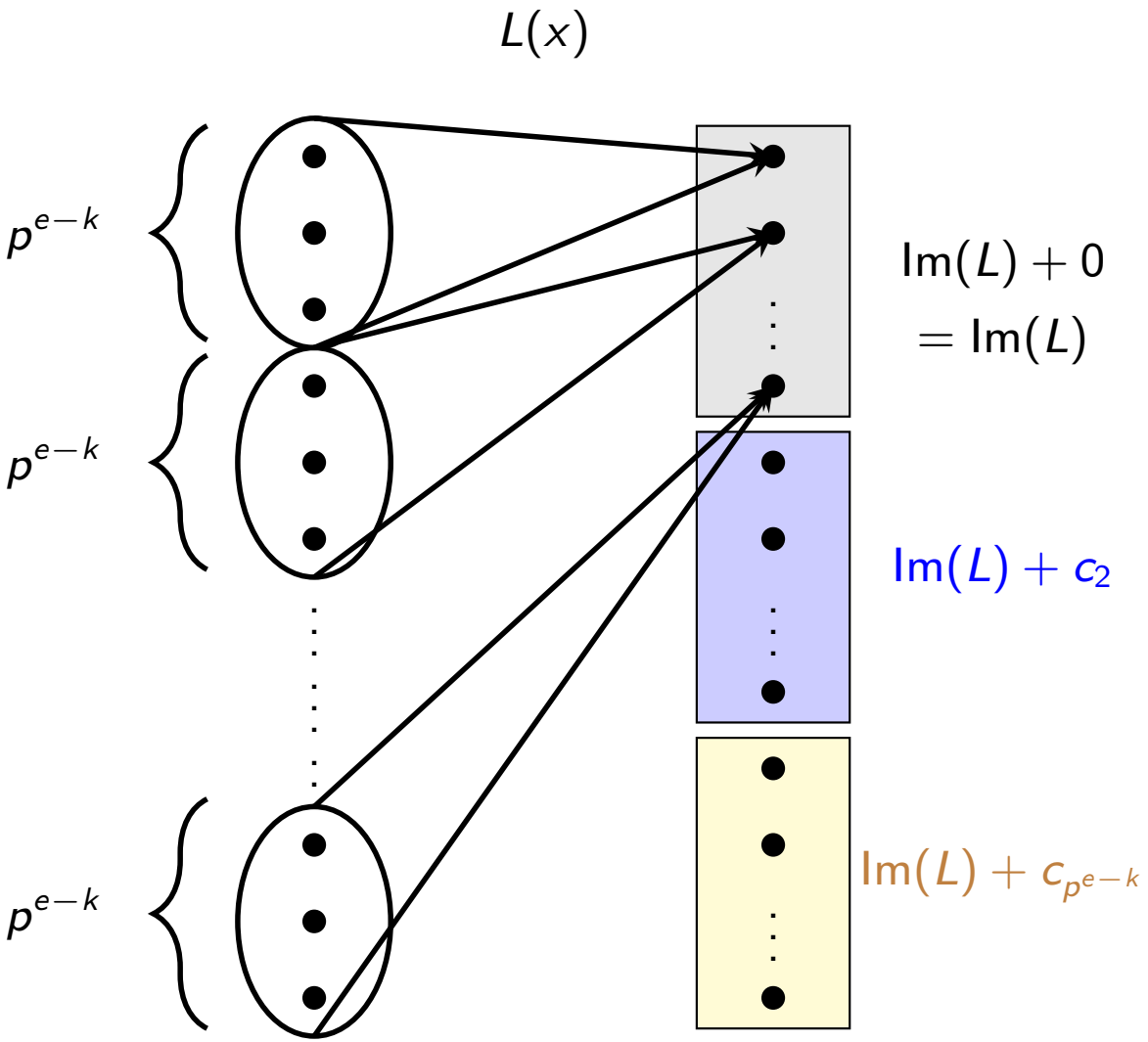

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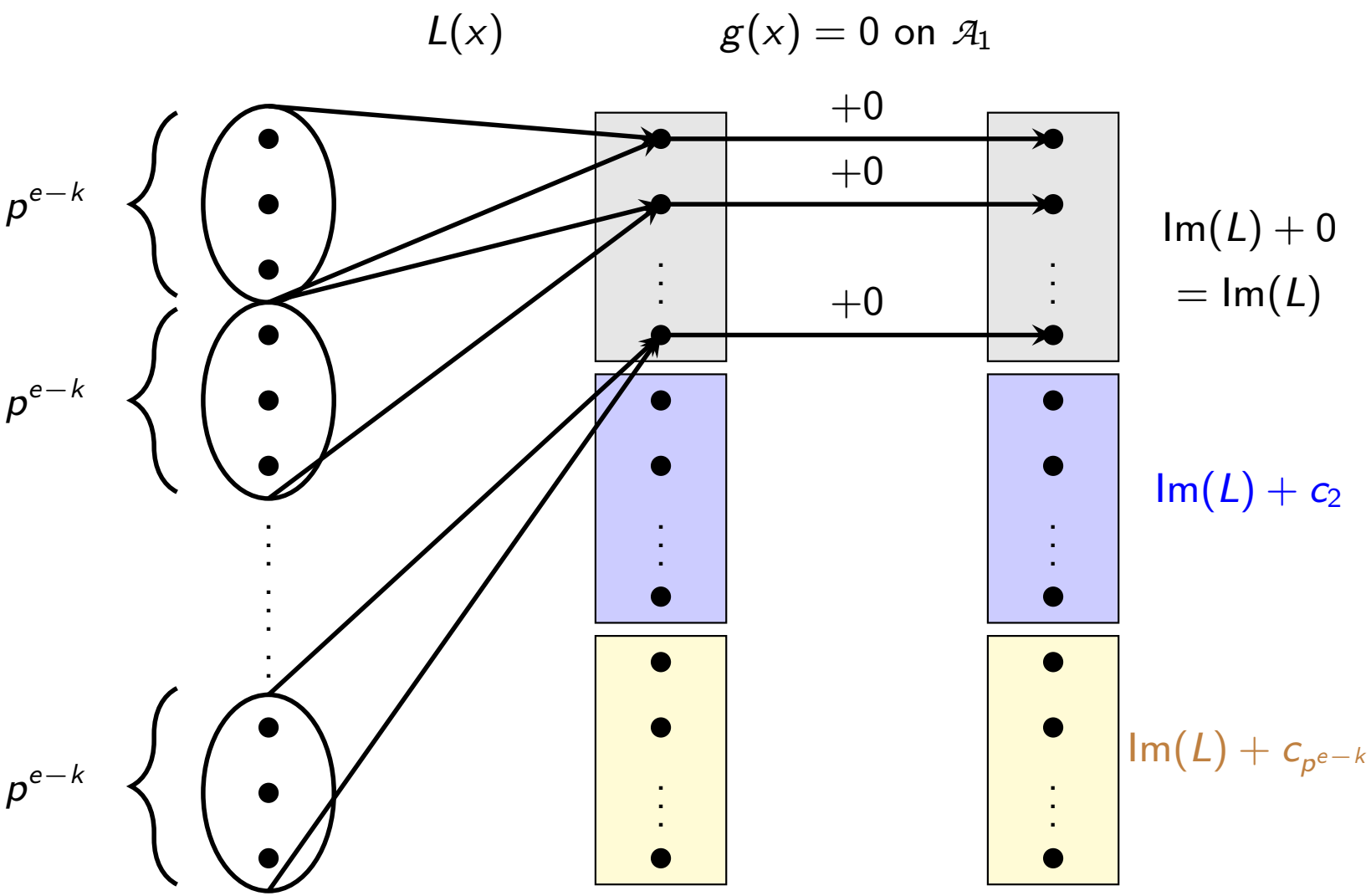

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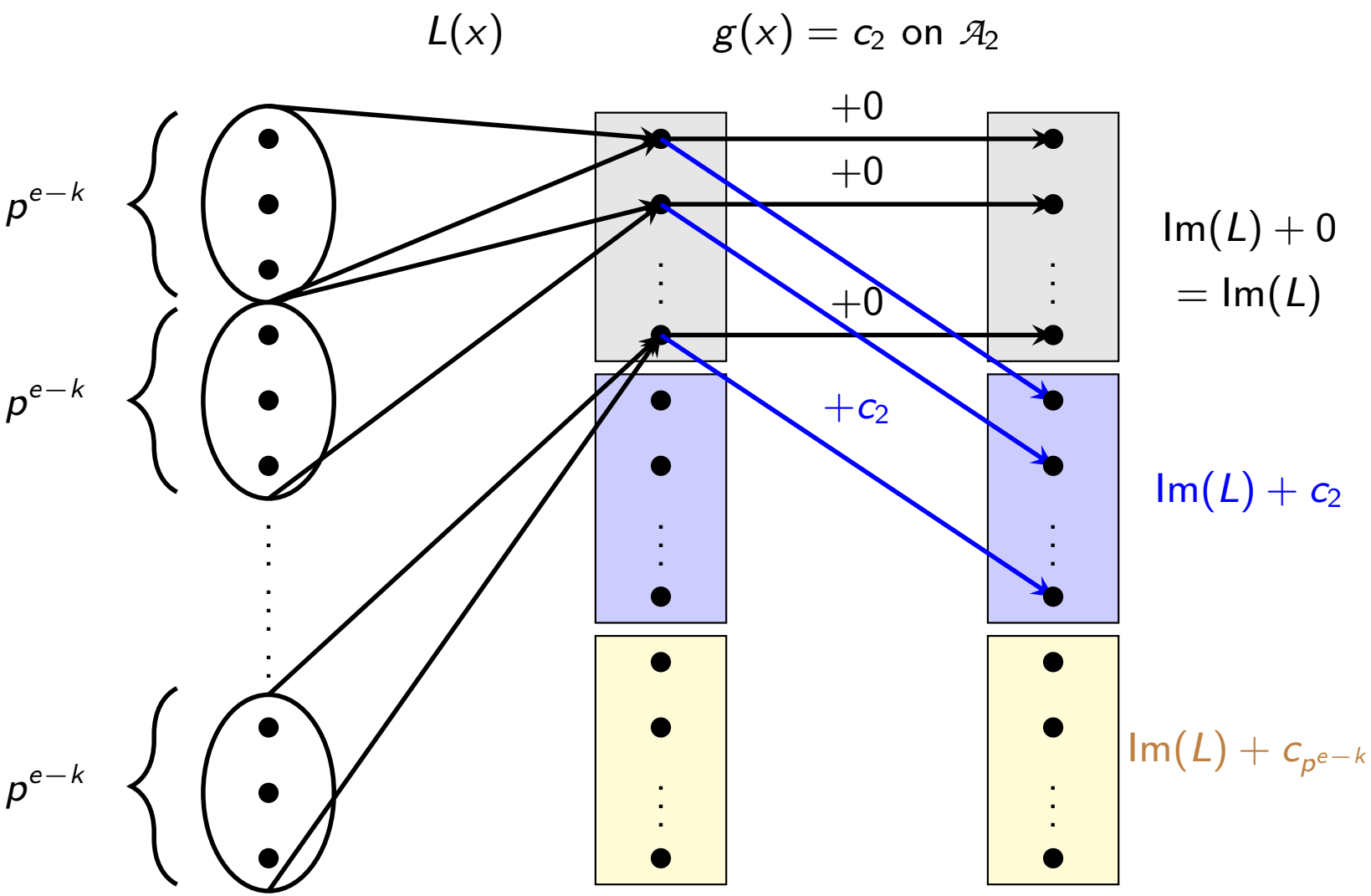

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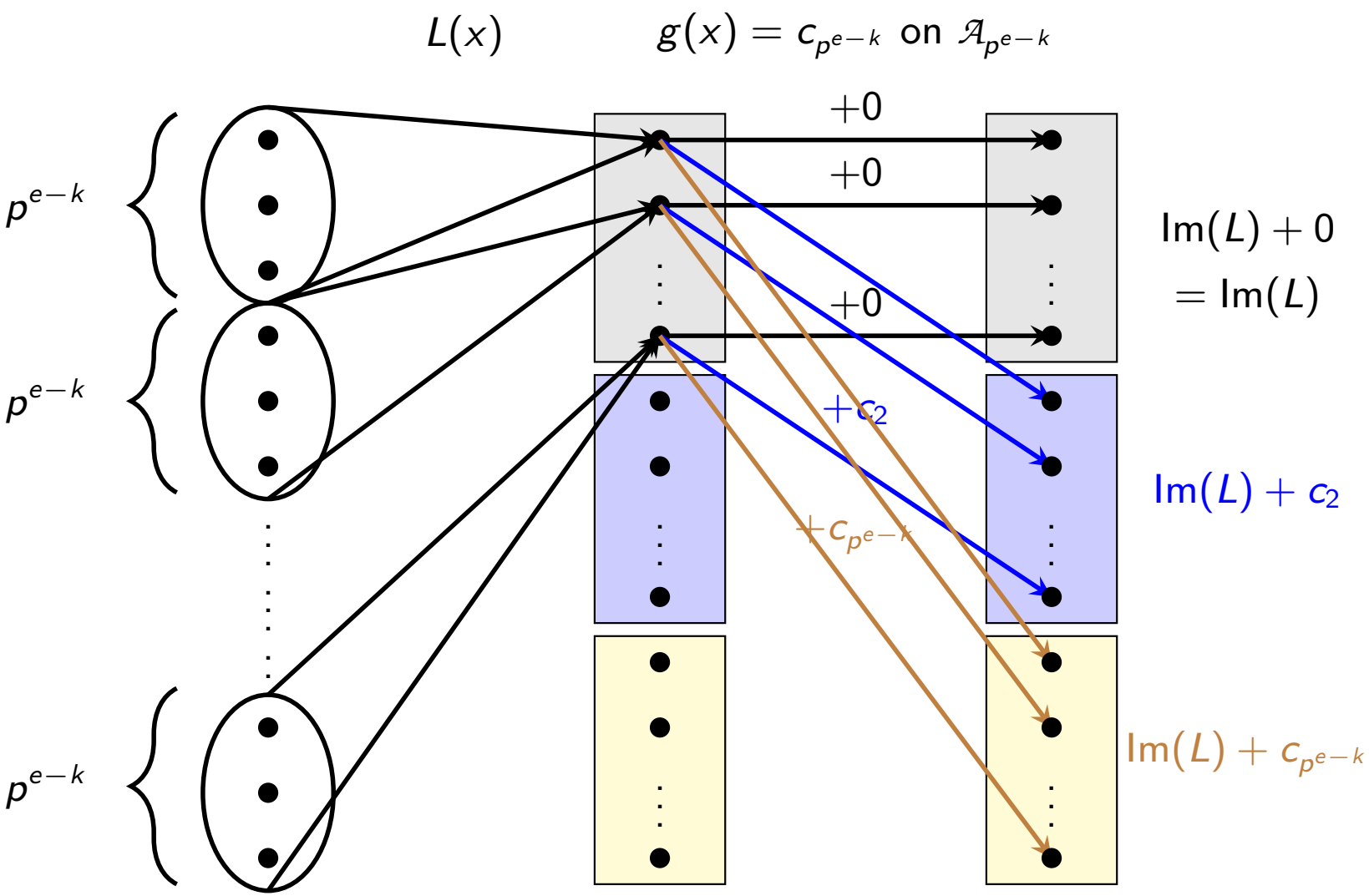

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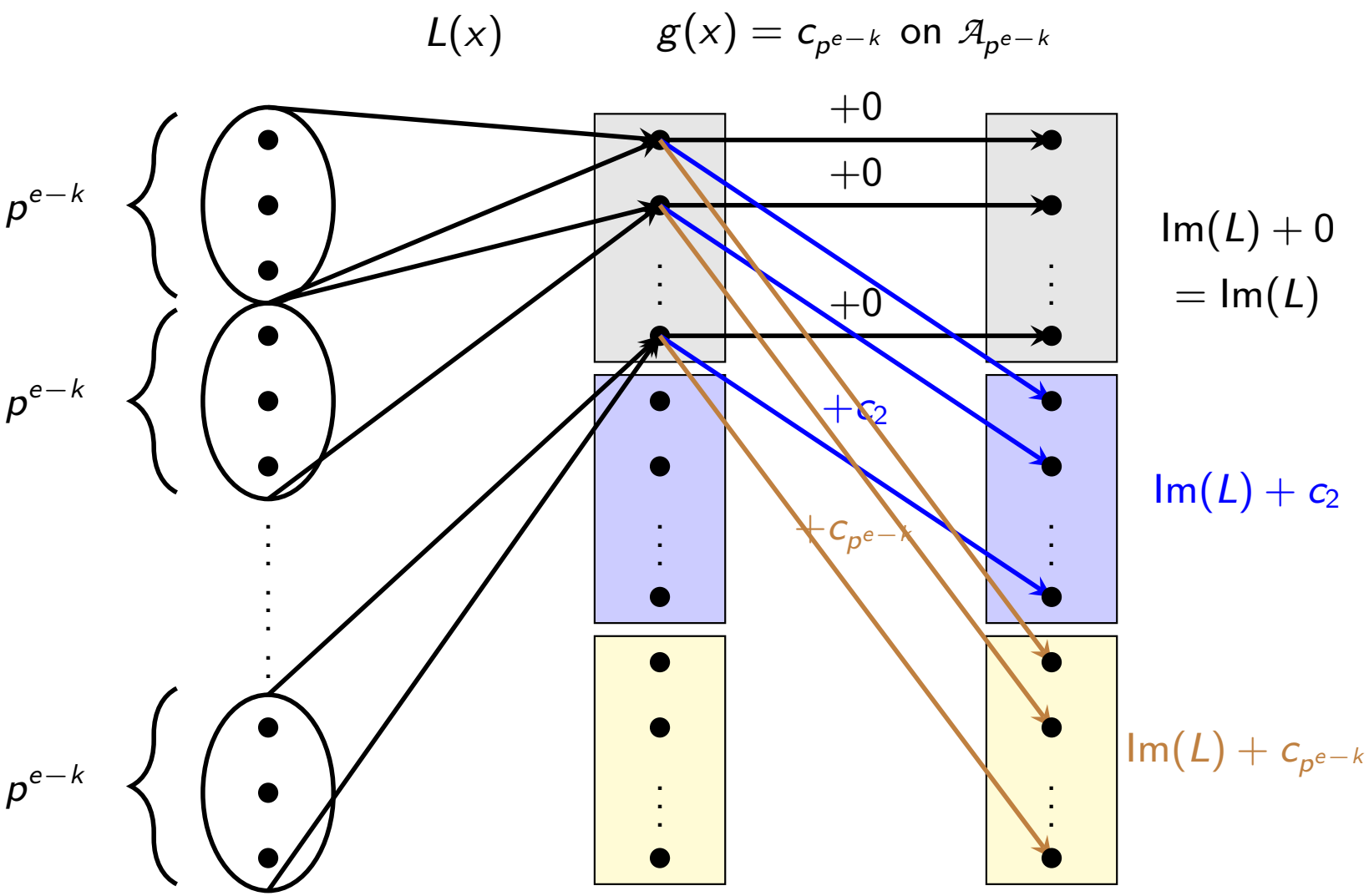

So $u(L) \leq \text{P-Res}(L) \leq p^{e-k} = u(L)$.

An application of permutation resemblance

Differential uniformity

Let $\mathcal{G}$ be an abelian group and $f : \mathcal{G} \to \mathcal{G}$.

Define the *differential operator* of f with respect to $a \in \mathcal{G}$ by

$$\Delta_{f,a}(x) = f(x+a) - f(x).$$

For $(a, b) \in \mathcal{G}^\star \times \mathcal{G}$, define

$$\delta_f(a, b) = \#\,\mathrm{PreIm}(\Delta_{f,a}, b).$$

The *differential uniformity of f (DU)* is given by

$$\delta_f = \max_{a \in \mathcal{G}^*, b \in \mathcal{G}} \delta_f(a, b).$$

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  And we're not doing very well.
- When q is odd, we have planar functions, which are 1-DU.
  Here, we've got a bigger issue, as we know planar functions cannot be permutations, meaning the best we can hope for is near-optimal DU permutations.

P-Res and DU

Theorem

Let $\mathcal{G}$ be a finite abelian group and $f, g : \mathcal{G} \to \mathcal{G}$. Then

$$\delta_{g+f} \leq \delta_f \cdot \left(V(g)^2 - V(g) + 1\right).$$

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Again, the bound comes from a worst-case scenario we don't expect to happen in most cases.

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This means $V(g) = \text{P-Res}(f)$, and $g + f \in \Omega_{\mathcal{G}}$. So

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We are therefore constructing permutations from f whose differential uniformity is bounded above by only δ_f and P-Res(f).

In the case where we start with a planar function, we find we are guaranteed to construct permutations $h = g + f$ for which

$$\delta_h \leq \text{P-Res}(f)^2 - \text{P-Res}(f) + 1.$$

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Theorem

Let $f \in \mathbb{F}_q[X]$, q odd, and suppose $f(0) = 0$ and f is two-to-one on $\mathbb{F}_q^\star$. Then

$$\text{P-Res}(f) \leq \left\lceil 2\sqrt{q-1} \right\rceil - 1.$$

When $q-1$ is a perfect square, the bound can be improved to

$$\text{P-Res}(f) \leq 2\sqrt{q-1} - 2.$$

Computational aspects

You really can!

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- permutations with the best possible DU over a given field (or even specifying a desired maximum DU).

The two algorithms can also be combined in such a way as to significantly reduce the number of variables of the combined IP while finding low DU functions among those permutations that most resemble a given f. The cost is that you can no longer insist upon optimal low DU.

Using the P-Res algorithm on x^2

All of the computational results I give here were generated on a simple laptop.

Our initial concerns with resemblance have been to prove some theoretical results and establish the feasiblility of computational results.

Using the P-Res algorithm on x^2

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Prime p	P-Res(x^2)	$\lceil 2\sqrt{q-1} \rceil - 1$
13 to 37	4	6 to 11
41	5	12
43,47,103	4	12,13,20
53 to 101	5	14 to 19
107 to 251	5	20 to 31
257	6	31
263,269,271	5	32
277,281	6	33
293,307,311	5	34,34,35
313	6	35
317	5	35
331,337	6	36

Note the very slow growth of P-Res.

We obtained similar results for prime powers $q \leq 343$, and for x^d with $d \mid (q-1)$.

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- Not surprisingly, for larger q, we start to run into memory and time issues.

The optimal DU IP algorithm is sufficiently adaptable that we can weaken the optimality condition, insisting only that the algorithm find a permutation with DU≥ 3.

When we do so, the algorithm finds 3-DU permutations over $\mathbb{F}_q$ for all $q \leq 37$ in decent time frames.

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- Over $\mathbb{F}_{64}$, the APN function x^3 has P-Res $= 7$. It took under 2 seconds to find a 6-DU permutation among those that resemble x^3 the closest. (This may or may not be optimal.)
  For the APN function $f(x) = x^{24} + \alpha^{59}x^{17} + \alpha^{60}x^3$, which has P-Res $= 5$, we find an optimal 6-DU permutation in under 3 minutes.

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- Over $\mathbb{F}_{256}$, again using the APN function x^3, with a non-optimal setting the algorithm finds an 8-DU permutation in under 4 minutes.

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Conjecture

There exists a 2-DU permutation over all sufficiently large finite fields.

Many thanks for your time.