On the Multiplicative Complexity of Cubic Boolean Functions

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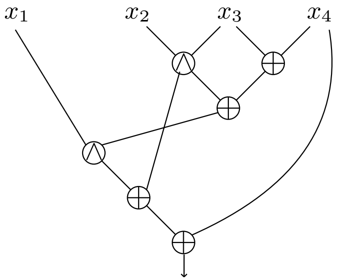
Overview

- Circuit optimization problem
- Multiplicative complexity
- Low MC circuits for cubic Boolean functions

Boolean Circuits

A *Boolean circuit* with n inputs and m outputs is a **directed acyclic graph** (DAG), where

- the inputs and the gates are *nodes*,
- the edges correspond to Boolean-valued *wires*,
- *fanin*/*fanout* of a node is the number of wires going in/out the node,
- the nodes with fanin zero are called *input nodes*
- a node with fanout zero is an *output node*



Boolean Function Complexity

Problem: Given a basis of Boolean gates, construct a circuit that computes a function that is optimal w.r.t. some criteria, such as

- Size: The number of gates in the circuit.
- Depth: The length of the longest path from an input gate to the output gate.

Almost all Boolean functions are complex.

Target metric depends on the application.

- Circuits with small number of gates use less energy and occupy smaller area, and are desired for *lightweight cryptography applications* running on constrained devices.
- Circuits with small number of AND gates are desired for *secure multi-party computation*, *zero-knowledge proofs* and *side channel protection*.
- Circuits with small AND-depth are desired for homomorphic encryption schemes.

Multiplicative Complexity (MC)

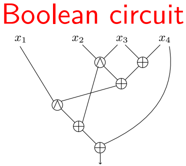
Minimum number of nonlinear gates needed to implement f by a Boolean circuit

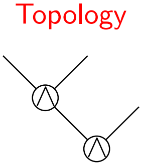
- Min. $\#$ of `AND` gates needed over the basis `(AND, XOR, NOT)`.
- Almost all $f \in B_n$ have MC at least $2^{n/2} - n - 1$ with high probability.
- No specific n-variable function had been proven to have MC larger than n.
- MC of a function with degree d is at least $d - 1$ (degree bound).
- The number of n-variable Boolean fucntions with MC k is at most $2^{k^2-k+2kn+n+1}$
- MC is affine invariant.
  - Boolean functions $f, g \in B_n$ are affine equivalent if there exists a transformation of the form $f(x) = g(Ax + a) + b \cdot x + c$, where $A \in GL(n, 2)$; $a, b \in \mathbb{F}_2^n$, and $c \in \mathbb{F}_2$.
  - The set of affine equivalent functions constitute an equivalence class denoted by $[f]$, where f is an arbitrary function from the class.
  - Affine equivalent Boolean functions have the same MC.

MC of Boolean Functions

Exhaustively construct all Boolean topologies with 1,2, 3, . . . AND gates, and evaluate the topologies until a function from $[f]$ is generated.

- Topology: Abstraction of a Boolean circuit that shows the relations between AND gates



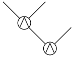


Constructing Topologies [CTP18]

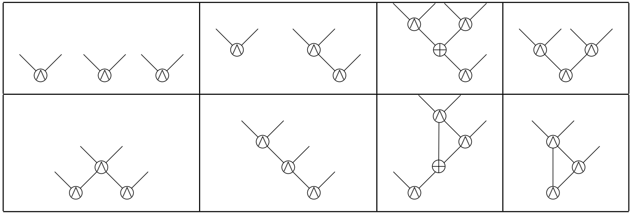
Topologies with 1 AND gate

Topologies with 2 AND gates

and



Topologies with 3 AND gates



Number of topologies with 4 AND gates is 84.

Finding the MC of Boolean functions

Different ways of determining the MC of a Boolean function

- Show that it is affine equivalent to a function whose MC is known.
- Find a circuit that satisfies a lower bound (degree bound).
- Iteratively construct all circuits with increasing #ANDs until the function is generated.

Solved up to 6-variables

- $C_\wedge(f) \leq n-1$ for $f \in B_n, n \leq 5$ (Turan, Peralta, 2014)
- $C_\wedge(f) \leq 6$ for $f \in B_6$ (Çalık et al., 2018)

Known methods are infeasible for $n > 6$, due to the large number of affine equivalence classes and Boolean circuits.

Boolean functions with MC 1 and 2

Boolean functions with MC 1 [FP02]

- Functions with MC 1 are affine equivalent to x_1x_2.
- The number of n-variable Boolean functions with MC 1 is $2\binom{2^n}{3}$.

Boolean functions with MC 2 [FTT17]

- Functions with MC 2 are affine equivalent to one of the functions from the set $\{x_1x_2x_3,\ x_1x_2x_3 + x_1x_4,\ x_1x_2 + x_3x_4\}$.
- The number of n-variable Boolean functions with MC 2 is

$$2^n(2^n-1)(2^n-2)(2^n-4)\left(\frac{2}{21}+\frac{2^n-8}{12}+\frac{2^n-8}{360}\right).$$

MC of Quadratic Boolean Functions

- The equivalance classes for n-bit quadratic Boolean functions are
  - x_1x_2 (with MC 1)
  - $x_1x_2 + x_3x_4$(with MC 2)
  - $x_1x_2 + x_3x_4 + x_5x_6$ (with MC 3)
  - $\ldots$
  - $x_1x_2 + x_3x_4 + \ldots + x_{n-1}x_n$ (even n) (with MC $\lfloor \frac{n}{2} \rfloor$)
  - $x_1x_2 + x_3x_4 + \ldots + x_{n-2}x_{n-1}$ (odd n) (with MC $\lfloor \frac{n}{2} \rfloor$)

- MC of a quadratic Boolean function is at most $\lfloor \frac{n}{2} \rfloor$.

Dimension of a Boolean function

The following functions are all affine equivalent and have MC=1:

$$x_1x_2$$
$$x_1 + x_2x_3$$
$$(x_1 + x_2)(x_3 + x_4) = x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4$$

It is easier to work on smaller number of variables.

Definition. Let L_f be the number of input variables that appear in the *algebraic normal form* (ANF) of a Boolean function f. The dimension of f is the smallest number of variables that appear in the ANF among the functions that are affine equivalent to f:

$$dim(f) = \min_{g \in [f]} L_g.$$

Example. $dim(x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4) = dim(x_1x_2) = 2$

A New MC Lower Bound based on Dimension

Theorem

For $f \in B_n$, $C_\wedge(f) \geq \lceil \mathrm{dim}(f)/2 \rceil$.

Sketch of the proof.

1. Let $C_\wedge(f) = k$, consider a circuit implementing f with k AND gates.
2. The topology with k AND gates has $2k$ linear function inputs.
3. The rank of $2k$ linear functions can be at most $2k$.
4. Any set of $2k$ linear functions on $n > 2k$ variables can be affine transformed to functions having at most $2k$ variables.
5. Therefore, $dim(f) \leq 2k$, which implies $C_\wedge(f) \geq \lceil dim(f)/2 \rceil$.

Example. Let $f = \Sigma_4^8 = x_1x_2x_3x_4 + \ldots + x_5x_6x_7x_8$. According to the degree bound, $C_\wedge(f) \geq 3$. By dimension bound, $C_\wedge(f) \geq 8/2 = 4$.

Multiplicative Complexity of Cubic Boolean Functions

The following results follow from earlier studies [CTP19, CTP18, TP14, FTT17]

- Let $f \in B_n$ be a Boolean function with MC 2. Then f is affine equivalent to exactly one of the following two functions: $x_1x_2x_3$ and $x_1x_2x_3 + x_1x_4$.
- Let f be an n-variable cubic Boolean function with dimension 5 and MC 3. Then f is affine equivalent to exactly one of the following four functions $x_1x_3x_4 + x_1x_2x_5$, $x_1x_2x_3 + x_4x_5$, $x_3x_4 + x_1x_3x_4 + x_1x_2x_5$ and $x_1x_2x_3 + x_2x_4 + x_1x_5$.
- Let f be an n-variable cubic Boolean function with dimension 6 and MC 3. Then f is affine equivalent to exactly one of the following three functions $x_3x_4 + x_1x_3x_4 + x_1x_2x_5 + x_1x_6$, $x_1x_3x_4 + x_1x_2x_5 + x_1x_6$ and $x_1x_2x_3 + x_4x_5 + x_1x_6$.

Construction Circuits for Cubic Boolean Functions

1. Decompose n-bit cubic Boolean function f such that

$$f = x_n f_1 + f_2$$

   where f_1 is a quadratic function defined on $(x_1, \ldots, x_{n-1})$ and f_2 is a function of degree at most three defined on $(x_1, \ldots, x_{n-1})$.
2. Optimally implement f_1 (with at most $\lfloor \frac{n-1}{2} \rfloor$ `AND` gates).
3. If f_2 is cubic, apply this method recursively. If not cubic, optimally implement f_2 (with at most $\lfloor \frac{n-1}{2} \rfloor$ `AND` gates)
4. Given the implementations of f_1 and f_2, implement f using one additional `AND` gate.

Upper bounds on the MC of Cubic Boolean Functions

The method provides an upper bound on the MC of n-variable cubic Boolean functions, denoted $\text{MaxMC}(B_n^c)$, using the following relation

$$\text{MaxMC}(B_n^c) \leq \text{MaxMC}(B_{n-1}^c) + \lfloor \frac{n-1}{2} \rfloor + 1. \tag{1}$$

We experimentally showed that MC of cubic Boolean function for $n = 7$ is at most 8.

$$\text{MaxMC}(B_n^c) \leq \frac{1}{2}(\lfloor \frac{n-1}{2} \rfloor^2 + \lfloor \frac{n-1}{2} \rfloor + (\lfloor \frac{n}{2} \rfloor - 1)\lfloor \frac{n}{2} \rfloor + 2(n-8)). \tag{2}$$

Upper bounds

Table: Upper bounds on the MC of n-variable Boolean functions

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Cubic functions	-	2	2	4	5	8	12	16	20	25	30	36	41	48	54
All functions	1	2	3	4	6	13	26	41	57	88	120	183	247	374	502

Upper bounds

The $\#$ of functions in B_n with MC $\leq k$ is bounded above by $2^{k^2+2kn+n+2}$.
$|B_n^c| = (2^{\binom{n}{3}} - 1)2^{\binom{n}{2}+n+1}$.
Let $\tau = \text{MaxMC}(B_n^c)$, we have

$$\begin{aligned}(2^{\binom{n}{3}} - 1)2^{\binom{n}{2}+n+1} &\leq 2^{\tau^2+2\tau n+n+2} \\ \binom{n}{3} + \binom{n}{2} + n &\leq \tau^2 + 2\tau n + n + 2 \\ n^3 - n &\leq 6\tau^2 + 12\tau n + 12\end{aligned}$$

$$\frac{\sqrt{6}}{6}(n^3 + 6n^2 - n - 12)^{\frac{1}{2}} - n \leq \tau, \tag{3}$$

which shows that $\text{MaxMC}(B_n^c)$ is $\Omega(n^{3/2})$. Thus

$$\Omega(n^{3/2}) \leq \text{MaxMC}(B_n^c) \leq O(n^2). \tag{4}$$

Closing this gap is an interesting open problem.

Conclusion

- Studied the MC of cubic Boolean functions
- Enumerated the exhaustive list of equivalence functions with MC ≤ 4.
- Presented a method to implement cubic Boolean functions that decomposes the function into an expression of functions defined on smaller number of variables.
- Provided an upper bounds on the MC of cubic Boolean functions, significantly better than the upper bounds for random Boolean functions.

More Information

- **NIST Circuit Complexity Project Webpage:**
  https://csrc.nist.gov/Projects/Circuit-Complexity
- **GitHubLink:**
  https://github.com/usnistgov/Circuits/
- **Contact email:**
  circuit_complexity@nist.gov

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