Recent results on the nonlinearity of Boolean functions

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Outline

Asymptotic results on the nonlinearity of

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Asymptotic results on the nonlinearity of

Boolean functions and more general functions

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Nonasymptotic results

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Asymptotic results on the nonlinearity of Boolean functions and more general functions

Nonasymptotic results

Autocorrelations of Boolean functions

Nonlinearity of Boolean functions

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$$f(x, y, z) = xy + yz + z$$

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The Hamming distance of f to the 16 affine Boolean functions is either 2, 4, or 6. Therefore the nonlinearity of f is 2.

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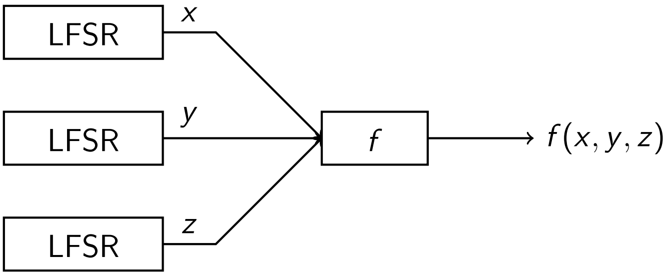

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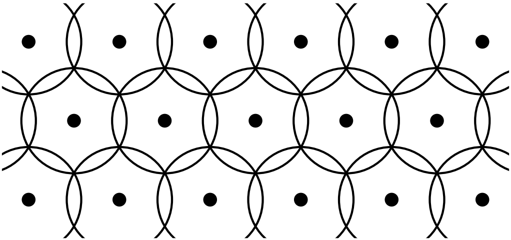
A related question

What is the largest nonlinearity of a balanced Boolean function on $\mathbb{F}_2^n$?

Coding theory

The covering radius of a code $\mathcal{C} \subseteq \mathbb{F}_2^N$ is the smallest number r, such that the spheres of radius r centred at the points of $\mathcal{C}$ cover the whole space $\mathbb{F}_2^N$.



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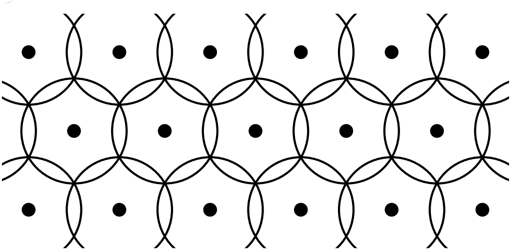

Main question (restated)

What is the covering radius of the first order Reed-Muller code $R(1, n)$?

Fourier transforms

The Fourier transform of $f : \mathbb{F}_2^n \to \mathbb{F}_2$:

$$\hat{f}(a) = \frac{1}{2^{n/2}} \sum_{y \in \mathbb{F}_2^n} (-1)^{f(y)} \, (-1)^{\langle a,y \rangle}.$$

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The nonlinearity of f equals $2^{n-1} - \mu(f)\, 2^{n/2-1}$, where

$$\mu(f) = \max_{a \in \mathbb{F}_2^n} |\hat{f}(a)|$$

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What is the smallest spectral radius $\mu(n)$ of a Boolean function on $\mathbb{F}_2^n$?

Parseval's identity

Parseval's identity is

$$\sum_{a \in \mathbb{F}_2^n} \hat{f}(a)^2 = 2^n,$$

so that the spectral radius of a Boolean function is at least 1 and the covering radius of $R(1, n)$ is at most $2^{n-1} - 2^{n/2-1}$.

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- $\mu(n) \leq \frac{7}{8}\sqrt{2} = 1.23\ldots$ for all $n \geq 9$ (Kavut-Yücel 2010)
- $\mu(n) \leq \frac{27}{32}\sqrt{2} = 1.19\ldots$ for all $n \geq 15$
  (Patterson-Wiedemann 1983)

What did Patterson-Wiedemann do?

For a subgroup $H \leq \mathrm{GL}_n(\mathbb{F}_2)$ consider *H-invariant functions*:

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Functions $\mathbb{F}_{2^4} \to \mathbb{F}_2$ that are H-invariant for $H = \{1, \theta^5, \theta^{10}\}$:

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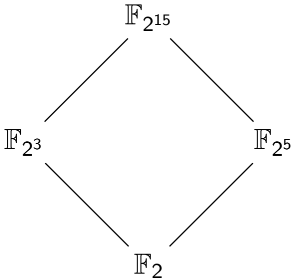
$\mathrm{GL}_{15}(\mathbb{F}_2)$ has a subgroup

$$H \cong \mathbb{F}_{2^3}^* \times \mathbb{F}_{2^5}^* \times \mathrm{Gal}(\mathbb{F}_{2^{15}}/\mathbb{F}_2)$$

of order $7 \cdot 31 \cdot 15 = 3255$.

This group partitions $\mathbb{F}_2^{15}$ into 10 orbits of size 3255 and one orbit of size 217.



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$$\tfrac{27}{32}\sqrt{2} = 1.1932\ldots.$$

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$\mathrm{Gal}(\mathbb{F}_{2^9}/\mathbb{F}_{2^3})$	176	≥ 242	$\leq 1.2374\ldots$

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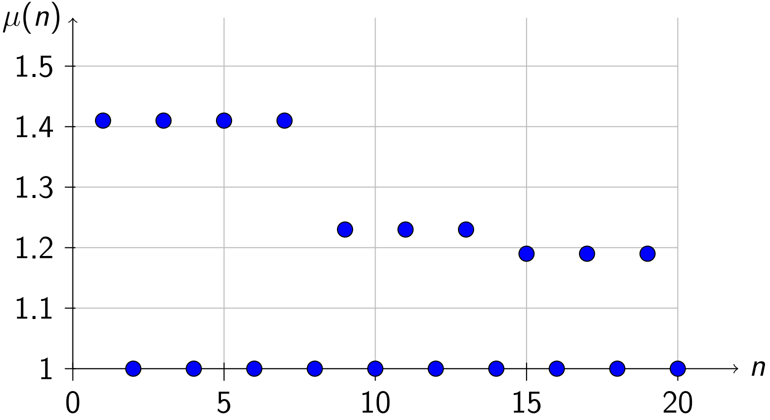
Best known nonlinearities



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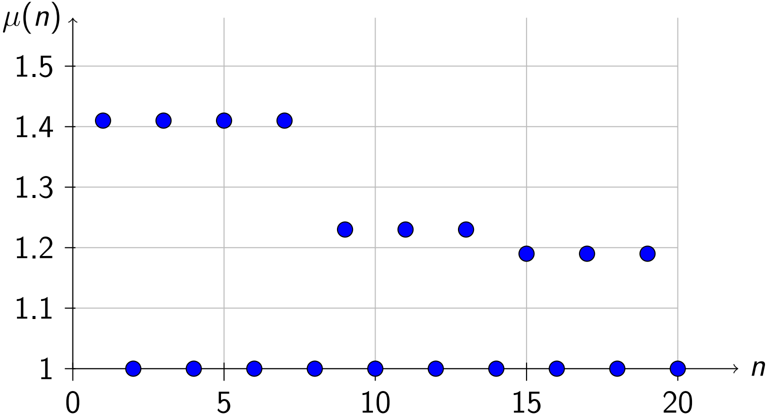
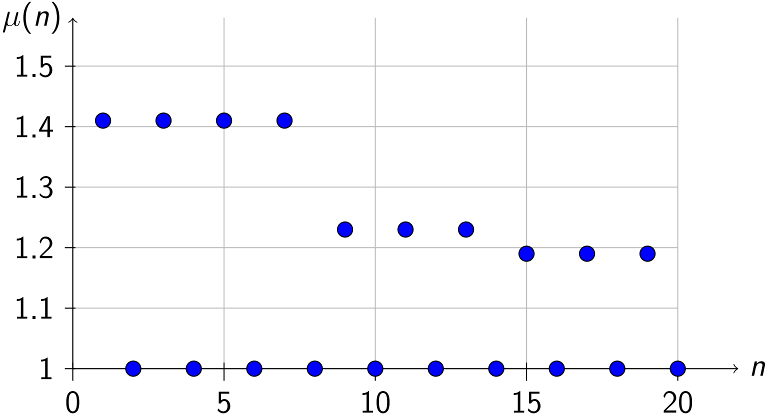


Conjecture (Patterson-Wiedemann 1983). $\lim_{n\to\infty} \mu(n) = 1$.

Best known nonlinearities



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Theorem (S. 2019). This conjecture is true.

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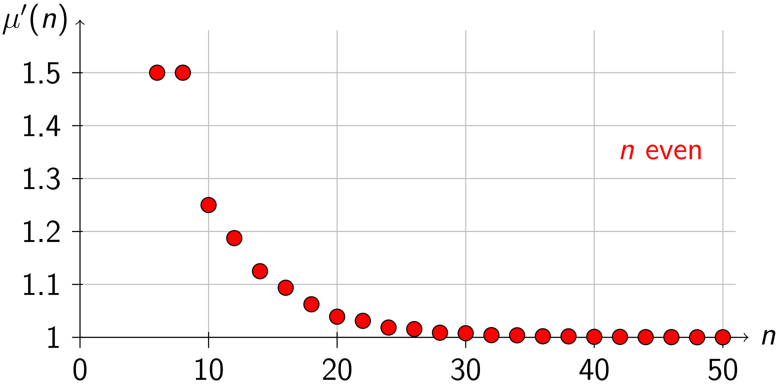

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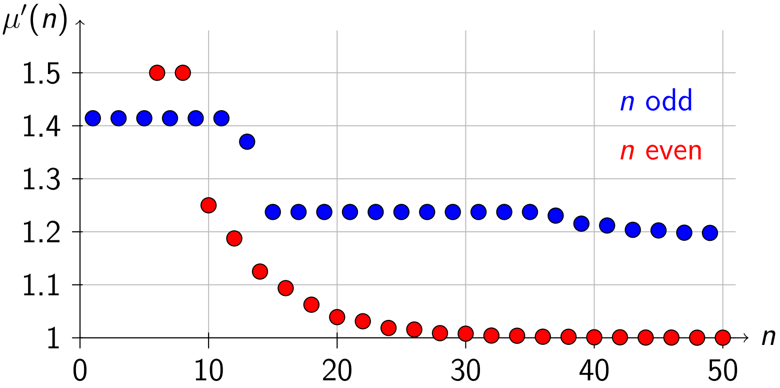

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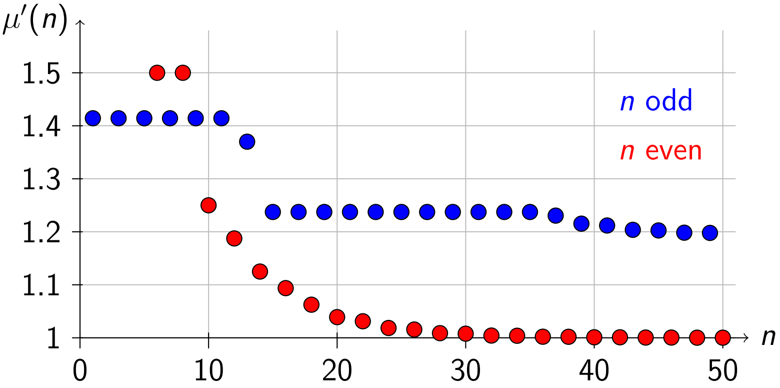

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The functions: An example

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Functions $\mathbb{F}_{2^4} \to \mathbb{F}_2$ with $H = \{1, \theta^5, \theta^{10}\}$:

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The main result follows by letting e tend to infinity.

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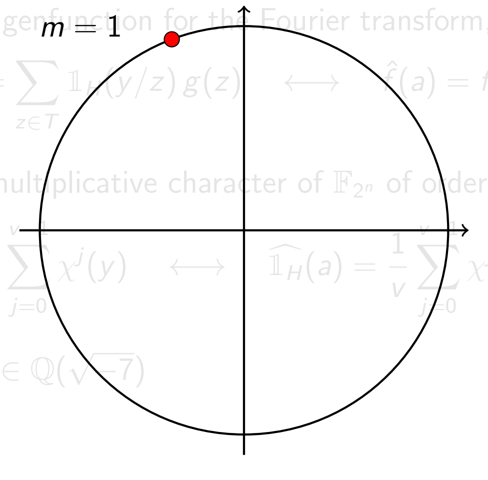

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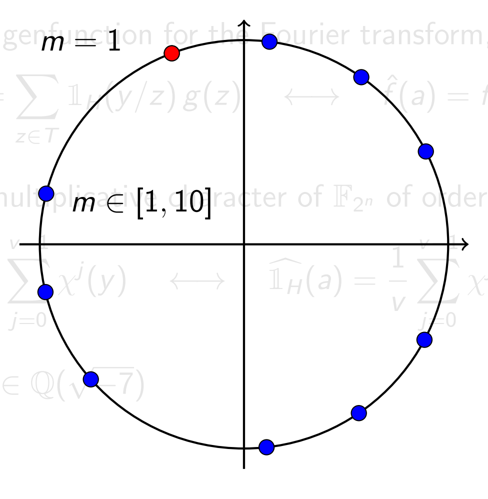

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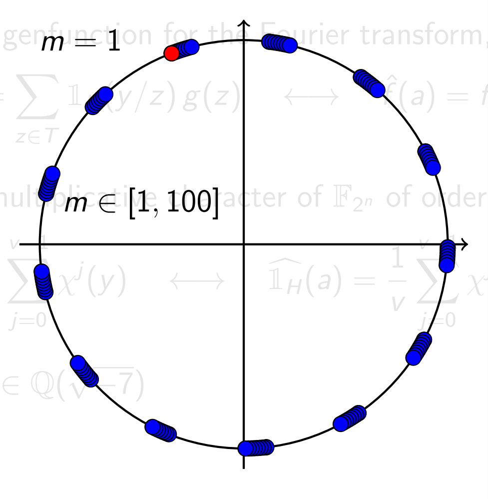

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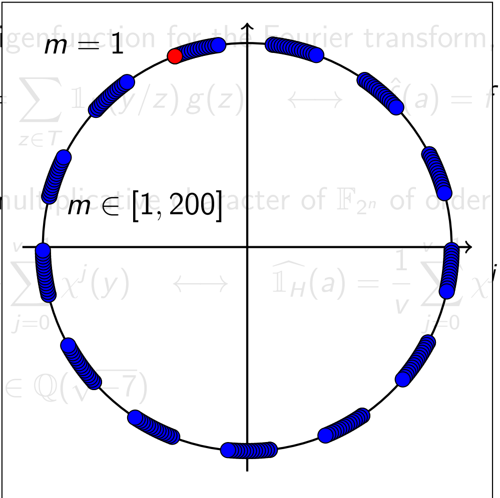

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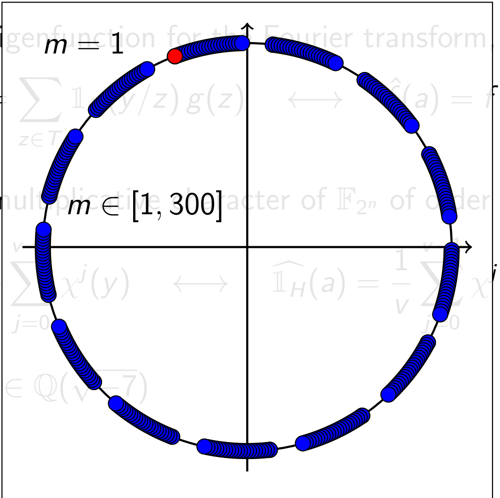

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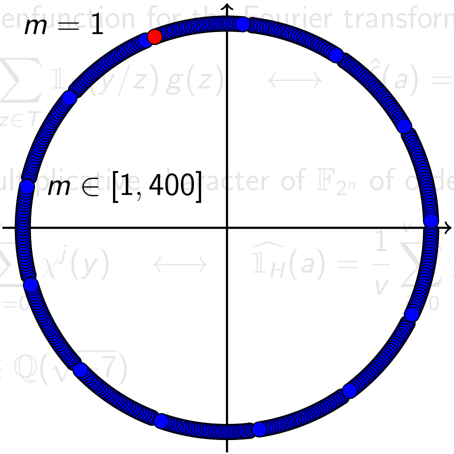

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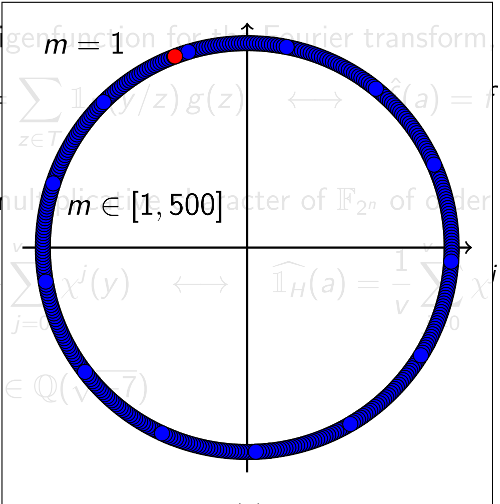

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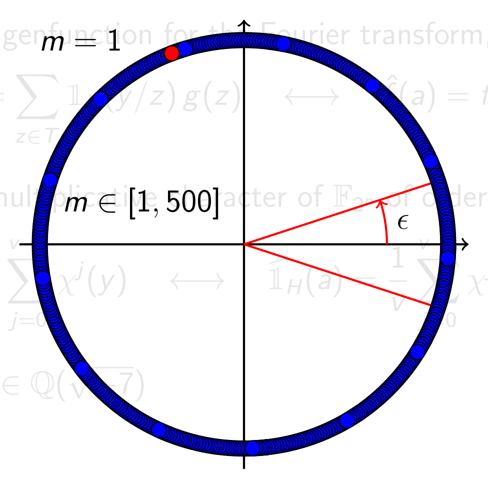

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Then $G(\chi^j) \in \mathbb{Q}(\sqrt{-7})$ and by the Davenport-Hasse Theorem

$$\frac{G(\chi^j)}{2^{n/2}} \approx 1 \quad \text{for some odd } n \text{ and all } 0 < j < v.$$

Six standard deviations suffice

Take an $M \times N$ matrix A with $M \geq N$ and real entries of magnitude at most 1.

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More general functions

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Equivalently, what is the covering radius of the generalised first order Reed-Muller code $R_q(1,n)$?

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The generalised Patterson-Wiedemann Conjecture

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$$\lim_{n\to\infty} \mu_q(n) = 1 \quad \text{for all prime powers } q.$$

Asymptotic nonlinearities

Theorem (S. 2020). Let q be a power of a prime p and suppose that there is another prime $r > 3$ such that $r \equiv 3 \pmod 4$ and $-p$ is a primitive root modulo r^2. Then $\lim_{n\to\infty} \mu_q(n) = 1$.

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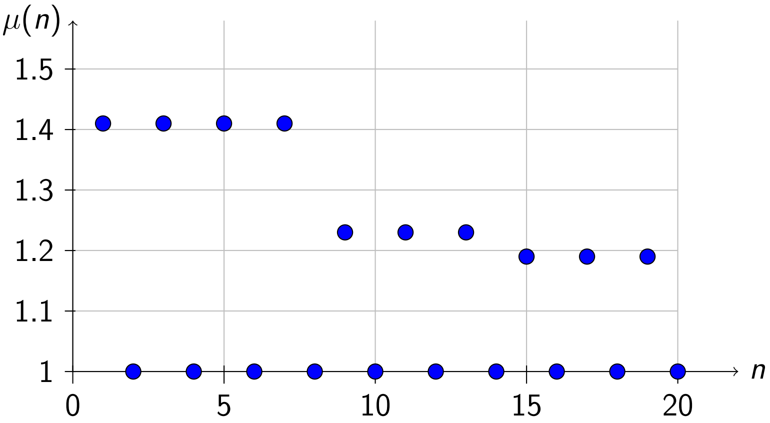
Theorem (S. 2020). Assume GRH. Then $\lim_{n\to\infty} \mu_q(n) = 1$ for all prime powers q.

Best known nonlinearities



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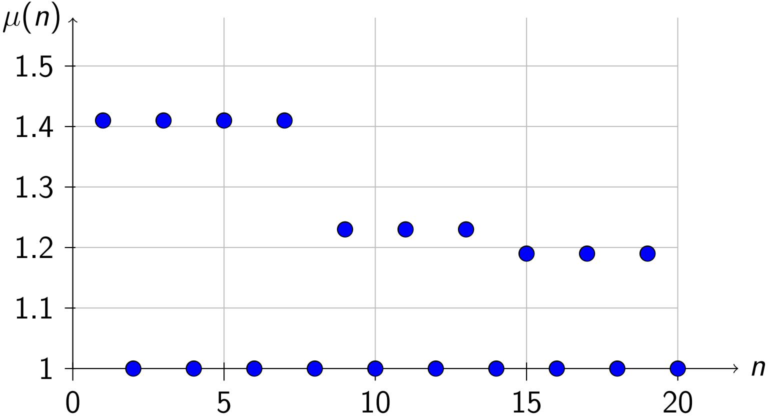

We now know that $\lim_{n\to\infty} \mu(n) = 1$,

however without improving any specific value of $\mu(n)$.

The quadratic residue construction

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Functions $\mathbb{F}_{2^4} \to \mathbb{F}_2$ with $H = \{1, \theta^5, \theta^{10}\}$, so that $v = 5$:

$$1^2 \equiv 1, \quad 2^2 \equiv 4, \quad 3^2 \equiv 4, \quad 4^2 \equiv 1 \pmod 5.$$

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In general, v is prime and $f(0) = 1$ and

$$f(y) = \mathbb{1}_H(y)h(y) + \sum_{k=0}^{v-1} \mathbb{1}_H(y/\theta^k)(k \,|\, v) \quad \text{for } y \in \mathbb{F}_{2^n}^*.$$

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We still have to choose $h : H \to \{-1, 1\}$.

The Fourier transform

Choose v prime, such that $v \equiv 3 \pmod 4$ and such that -2 is a primitive root modulo v. For example, $v = 7, 23, 47, 71, 79$.

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$$G(\chi) = a + b\sqrt{-v}$$

for some unique $a, b \in \mathbb{Z}$ satisfying $a^2 + vb^2 = 2^n$.

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The maximum modulus is close to 1 if and only if $\frac{G(\chi)}{2^{n/2}} \approx \pm 1$.

Improving Spencer's theorem

Take an $M \times N$ matrix A with $M \geq N$ and real entries of magnitude at most 1.

Theorem (Spencer 1985). For all sufficiently large N, there exists $u \in \{-1, 1\}^N$ such that

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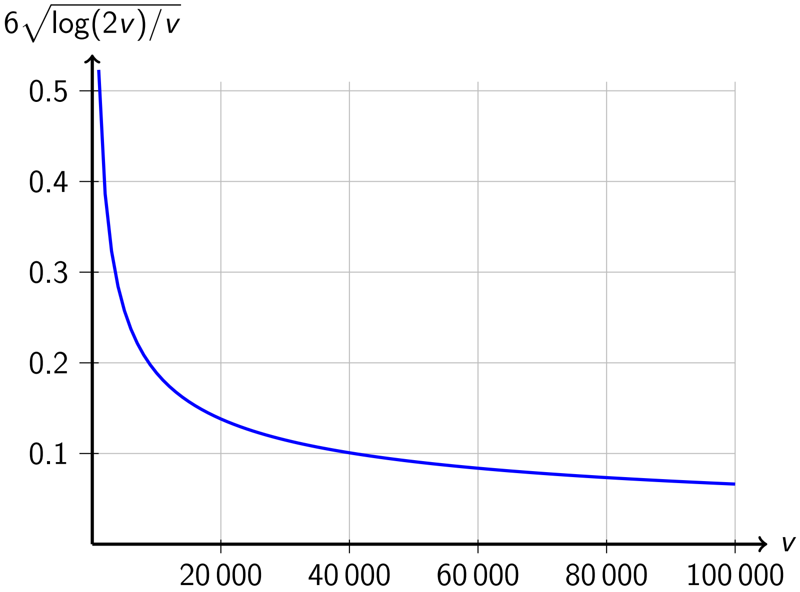
This shows the existence of $h : H \to \{-1, 1\}$ such that

$$f(y) = \mathbb{1}_H(y)h(y) \quad \longleftrightarrow \quad |\hat{f}(a)| \leq 6\sqrt{\log(2v)/v}.$$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0.4
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20 000
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v

Smallest known nonlinearities

- $\mu(3) = \sqrt{2}$ (easy to check)
- $\mu(5) = \sqrt{2}$ (Berlekamp-Welch 1972)
- $\mu(7) = \sqrt{2}$ (Mykkeltveit 1980), (Hou 1996)
- $\mu(n) \leq 1.237\ldots$ for all $n \geq 9$ (Kavut-Yücel 2010)
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Autocorrelations

The autocorrelation of a Boolean function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ at shift $u \in \mathbb{F}_2^n$ is

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A related question

What is the smallest absolute indicator $\delta(n)$ of a Boolean function on $\mathbb{F}_2^n$?

Small autocorrelations

The autocorrelations can be computed from the Fourier transform:

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The best known general result is (Zhang-Zheng 1996)

$$\delta(n) \le \sqrt{2} \quad \text{for all odd } n.$$

The Zhang-Zheng conjecture

(Zhang-Zheng 1996) constructed balanced Boolean functions f on $\mathbb{F}_2^n$ satisfying

$$\delta(f) \le \begin{cases} 2 & \text{for even } n \\ \sqrt{2} & \text{for odd } n. \end{cases}$$

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Conjecture (Zhang-Zheng 1996). For every balanced Boolean function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ we have

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This conjecture has been disproved for several small values of n by using the Patterson-Wiedemann approach together with heuristic search techniques.

Infinitely many counterexamples

Theorem (Tang-Maitra 2018). For each $n \geq 46$ with $n \equiv 2 \pmod 4$ there is a balanced function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ such that

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This gives counterexamples for all even $n \geq 20$.

Tweaking bent functions

Let $f : \mathbb{F}_2^n \to \mathbb{F}_2$ be a bent function. Then f is perfect:

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Show that this does typically not change $\delta(f)$ and $\mu(f)$ by much and that we typically get a nearly balanced function. Then only a few more bit flips make the function balanced.

New conjectures

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It seems that the functions used in the proof of the Patterson-Wiedemann Conjecture can be used to prove this.

Recent results on the nonlinearity of Boolean functions

Kai-Uwe Schmidt

Paderborn University