# Perturbation of Binary de Bruijn Sequences and the Spectral Theory of Graphs

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We continue an investigation of de Bruijn sequences initiated in [1]. In particular, we describe how to generate all de Bruijn sequences of a given order starting from a non-singular linear or nonlinear recursions. We realize this by the method of joining cycles and the method of cross-join pairs. The present results form a quantitative realization of the main theorem (Theorem 3.1) in [4].

Let $S(n)$ be the set of functions $\mathbb{F}_2^{n-1} \to \mathbb{F}_2$, which generate de Bruijn sequences of order $n$. For a function $F : \mathbb{F}_2^{n-1} \to \mathbb{F}_2$, we define the set $S(F;k)$ of functions $g \in S(n)$ such that the weight of the function $F + g$ equals $k$. The definition means that the number of inputs for which the functions $F$ and $g$ are different equals $k$. We use the notation $N(F;k) = |S(F;k)|$ and $G(F;y) = \sum_k N(F;k)y^k$. Let $\ell : \mathbb{F}_2^{n-1} \to \mathbb{F}_2$ be a linear function that generates an $m$-sequence of period $2^n - 1$. The following formula, due to Michael Fryers, was proved in [1]:

$$G(\ell;y) = \frac{1}{2^n}\left((1+y)^{2^{n-1}} - (1-y)^{2^{n-1}}\right).$$

We call $G(\ell;y)$ Fryers' polynomials. Their coefficients are generalization of the Helleseth and Kløve [3] formula for the number of cross-join pairs. We show how to calculate an analog polynomial when $\ell$ is a nonlinear Boolean feedback function of an *NLFSR* that generates a modified de Bruijn sequence. The corresponding Fryers' polynomial is found by determining the weighted adjacency matrix introduced in [1] and calculating its determinant. The matrix can be reduced to a simpler one by using the Hadamard matrix. In turn, the reduced matrix can be interpreted as adjacency matrix of some covering graph of the weighted de Bruijn graph. The obtained results are similar to those in [2].

## References

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